Sheaves of categories over topological and coaffine stacks

based on joint work with Nicolò Sibilla and James Pascaleff

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Plan of the talk

Notations and conventions

1. We fix a base field $\Bbbk$ of characteristic 0.
2. All notions appearing during the talk are to be interpreted in a derived (∞-categorical) sense: by "categories" we mean ∞-categories / dg categories; by "limits" and "colimits" we mean *homotopy* limits and colimits; by "$\Bbbk$-modules" we mean "chain complexes in the derived category of $\Bbbk$-modules"; by "tensor products" we mean *derived* tensor products; and so forth.
3. We use *homological* indexing conventions.

Sheaves and hypersheaves over topological spaces

Let X be a sufficiently nice topological space. We can define the category of sheaves over X with values in the category $\mathcal{S}$ of spaces as

$$\mathrm{Shv}(X) \coloneqq \mathrm{Shv}_\tau(\mathrm{Open}(X); \mathcal{S}).$$

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The category $\mathrm{Shv}^{\mathrm{hyp}}(X)$ is a localization of $\mathrm{Shv}(X)$:

1. It is a full subcategory (spanned by those objects for which equivalences can be detected at the level of stalks).
2. The inclusion $\mathrm{Shv}^{\mathrm{hyp}}(X) \subseteq \mathrm{Shv}(X)$ admits a hypercompletion left adjoint

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$$(-)^{\mathrm{hyp}} : \mathrm{Shv}(X) \longrightarrow \mathrm{Shv}^{\mathrm{hyp}}(X).$$

We say that a sheaf $\mathcal{F}$ is a *hypersheaf* if it belongs to $\mathrm{Shv}^{\mathrm{hyp}}(X)$.

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Consider the terminal morphism $\Gamma\colon X \to \{*\}$.

Definition

1. A hypersheaf is *hyperconstant* if it belongs to the essential image of the functor

$$\Gamma^{*,\mathrm{hyp}}\colon \mathcal{S} \simeq \mathrm{Shv}(\{*\}) \xrightarrow{\Gamma^*} \mathrm{Shv}(X) \xrightarrow{(-)^{\mathrm{hyp}}} \mathrm{Shv}^{\mathrm{hyp}}(X).$$

2. A hypersheaf is *locally hyperconstant* if there exists a covering U_α of X such that the restriction to each U_α is hyperconstant.

Local systems in homotopy theory

Warning

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Locally hyperconstant hypersheaves are the "correct" concept we need if we want to deal with local systems in homotopy theory.

Proposition

Let X be a topological space and $\Pi_\infty(X)$ its associated fundamental groupoid. There exists an equivalence of categories

$$\{\text{Locally hyperconstant hypersheaves on } X\} \simeq \mathrm{Fun}(\Pi_\infty(X), \mathcal{S}).$$

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For any cocomplete category $\mathscr{C}$, we shall write

$$\operatorname{LocSys}(X; \mathscr{C}) := \operatorname{Fun}(\Pi_\infty(X), \mathscr{C}).$$

Monodromy equivalence

If $\mathscr{C}$ is a cocomplete category, then the category of spaces acts on $\mathscr{C}$:

$$\begin{aligned} \mathcal{S}\otimes\mathscr{C} &\longrightarrow \mathscr{C} \\ (X, C) &\mapsto \operatorname*{colim}_X C. \end{aligned}$$

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For X a connected topological space and $\Omega_* X$ its based loop space

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Remark

When $\mathscr{C} = \mathrm{Mod}_{\Bbbk}$, this is the derived analogue of the fact that the abelian category of local systems of discrete $\Bbbk$-modules is the same as the abelian category of representations of the fundamental group $\pi_1(X)$.

Topological spaces as stacks

In derived algebraic geometry, we can encode the theory of homotopy types of topological spaces as constant stacks.

Definition

Let $\mathrm{Aff}_{\Bbbk} := \left(\mathrm{CAlg}_{\Bbbk}^{\geqslant 0}\right)^{\mathrm{op}}$ be the category of affine derived schemes, and let

$$\mathrm{St}_{\Bbbk} := \mathrm{Shv}_{\text{ét}}(\mathrm{Aff}_{\Bbbk}, \mathcal{S})$$

be the category of derived stacks for the étale topology.
The *Betti stack* X_{B} of a topological space X is the image of $\Pi_{\infty}(X)$ under the geometric morphism

$$\pi^* : \mathcal{S} \simeq \mathrm{Shv}_{\text{ét}}(\{*\}) \rightleftharpoons \mathrm{St}_{\Bbbk} : \pi_*.$$

Betti stacks, cohomology and local systems

Being stacks, to any Betti stack X_{B} we can associate both a commutative algebra of global sections $\Gamma(X_{\mathrm{B}}, \mathcal{O}_{X_{\mathrm{B}}})$ and a stable $\Bbbk$-linear and symmetric monoidal category $\mathrm{QCoh}(X_{\mathrm{B}})$ of quasi-coherent sheaves.

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Fact

1. *The algebra of global sections* $\Gamma(X_{\mathrm{B}}, \mathcal{O}_{X_{\mathrm{B}}})$ *agrees with the algebra of* $\Bbbk$*-cochains* $\mathrm{C}^\bullet(X;\Bbbk)$.
2. *For any affine scheme* $\mathrm{Spec}(R)$ *we have a symmetric monoidal equivalence*

$$\mathrm{QCoh}(X_{\mathrm{B}} \times \mathrm{Spec}(R)) \simeq \mathrm{LocSys}(X;\mathrm{Mod}_R) =: \mathrm{LocSys}(X;R).$$

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So Betti stacks provide a natural framework where to study cohomological properties of topological spaces.

Coaffine stacks

Betti stacks are highly pathological and difficult to work with, in practice. The notion of *affinization of homotopy types* due to Töen allows to consider some stacks which, even if far from being affine, still share some similarities with affine schemes and are in some sense the closest and best approximation of Betti stacks via less pathological objects.

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A *coaffine stack* X is any stack corepresented by a coconnective algebra A:

$$X \simeq \mathrm{Map}_{\mathrm{CAlg}_{\Bbbk}}(A, \Gamma(-, \mathcal{O})) \colon \mathrm{Aff}_{\Bbbk}^{\mathrm{op}} \longrightarrow \mathcal{S}.$$

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Notation

If X is coaffine and corepresented by A, we write $\mathrm{cSpec}(A) := X$.

Coaffine stacks: properties

Fact

1. *Every map from any stack $\mathcal{X}$ to a coaffine stack* $\mathrm{cSpec}(A)$ *is uniquely determined by a map of commutative algebras from A to* $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.

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$$\mathrm{QCoh}(\mathrm{cSpec}(A)) := \lim_{\substack{\mathrm{Spec}(R) \to \mathrm{cSpec}(A) \\ R \in \mathrm{CAlg}_{\Bbbk}^{\geqslant 0}}} \mathrm{Mod}_R$$

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Quasi-coherent sheaves on coaffine stacks

Theorem ([Lur11])

Let $X := \mathrm{cSpec}(A)$ *be a coaffine stack. Since* $\mathrm{H}_0(A) \cong \Bbbk$*, there exists a pointing* $\eta\colon \mathrm{Spec}(\Bbbk) \to X$.

1. *The category* $\mathrm{QCoh}(X)$ *admits a both left and right complete t-structure where an object* $\mathscr{F}$ *is (co)connective if and only if* $\eta^*\mathscr{F}$ *is (co)connective inside* $\mathrm{Mod}_{\Bbbk}$.

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2. *The category* Mod_A *admits a right complete t-structure where an object is coconnective if and only if the underlying* $\Bbbk$*-module is coconnective. Connective objects are those which become connective in the classical sense after base change along any map of commutative algebras* $A \to R$*, with* R *connective.*

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3. *The naturally defined functor* $F\colon \mathrm{Mod}_A \to \mathrm{QCoh}(X)$ *exhibits* $\mathrm{QCoh}(X)$ *as the left completion of the t-structure on* Mod_A.

Coaffine stacks and topological stacks

Summing up everything we said up to this point, given a topological space X we have a natural map

$$\mathrm{aff}\colon X_{\mathrm{B}} \longrightarrow \mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$$

induced by the identity map at the level of global sections$\mathrm{C}^\bullet(X;\Bbbk) \to \mathrm{C}^\bullet(X;\Bbbk) \simeq \Gamma(X_{\mathrm{B}}, \mathcal{O}_{X_{\mathrm{B}}})$.

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In the following, for any pointed stack $\eta\colon \mathrm{Spec}(\Bbbk) \to \mathcal{X}$ let $\mathrm{QCoh}(\mathcal{X})^{\mathrm{sm}}$ be the full subcategory of $\mathrm{QCoh}(\mathcal{X})$ spanned by those sheaves $\mathcal{F}$ whose pullback $\eta^*\mathcal{F}$ is a perfect object in $\mathrm{QCoh}(\mathrm{Spec}(\Bbbk)) \simeq \mathrm{Mod}_{\Bbbk}$.

Correspondence between quasi-coherent sheaves

Proposition ([Lur11; PPS24])

If X has the homotopy type of a connected CW complex which has a finite number of cells in each dimension, we have a diagram of stable categories

$$\begin{array}{ccc} \mathrm{Mod}_{\mathrm{C}^\bullet(X;\Bbbk)} & \xrightarrow{\simeq} & \mathrm{Ind}\left(\mathrm{QCoh}(X_{\mathrm{B}})^{\mathrm{sm}}\right) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) & \xrightarrow[\mathrm{aff}^*]{\simeq} & \mathrm{QCoh}(X_{\mathrm{B}}). \end{array}$$

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Remark

This recovers the classical Koszul duality correspondence between $\mathrm{IndCoh}^{\mathrm{L}}(\mathrm{C}_\bullet(\Omega_* X;\Bbbk)) := \mathrm{Ind}\left(\mathrm{LMod}^{\mathrm{sm}}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)}\right)$ and $\mathrm{Mod}_{\mathrm{C}^\bullet(X;\Bbbk)}$.

Reminders on presentable categories

Recall that a presentable category is a cocomplete category $\mathscr{C}$ with a small set of (compact) generators.

Fact

1. *There is a 2-category* $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ *having presentable categories as objects and (presentable) categories of colimit-preserving functors (or, equivalently, functors which are left adjoints) as morphisms.*

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3. *A presentable category* $\mathscr{C}$ *is* $\Bbbk$*-linear if it is a module for the presentably monoidal category* $\mathrm{Mod}_{\Bbbk}$ *inside* $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$*. This is equivalent to* $\mathscr{C}$ *being enriched over* $\mathrm{Mod}_{\Bbbk}$*.*

Sheaves of categories over prestacks

In [Gai15], Gaitsgory proposes the following categorification of the concepts of quasi-coherent sheaves and affineness.

Definition ([Gai15])

Let $\mathcal{X}$ be any (pre)stack. The 2-category of sheaves of categories over $\mathcal{X}$ is

$$\mathrm{ShvCat}(\mathcal{X}) := \lim_{\substack{\mathrm{Spec}(R)\to\mathcal{X} \\ R\in\mathrm{CAlg}_{\Bbbk}^{\geqslant 0}}} \mathrm{Mod}_{\mathrm{Mod}_R}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}.$$

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To any sheaf of categories $\mathcal{F}$ we can associate a $\operatorname{QCoh}(\mathcal{X})$-linear category

$$\Gamma^{\mathrm{enh}}(\mathcal{X},\mathcal{F}) := \lim_{\iota_R\colon \operatorname{Spec}(R)\to\mathcal{X}} \iota_{R,*}\Gamma(\operatorname{Spec}(R),\mathcal{F})$$

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and from any $\mathrm{QCoh}(\mathcal{X})$-linear category $\mathscr{C}$ one can obtain a sheaf of categories $\mathrm{Loc}_{\mathcal{X}}(\mathscr{C})$ by sheafifying the rule

$$\{\mathrm{Spec}(R)\to\mathcal{X}\}\mapsto \mathscr{C}\otimes_{\mathrm{QCoh}(\mathcal{X})}\mathrm{Mod}_R.$$

1-affineness

The functors $\mathrm{Loc}_{\mathcal{X}}$ and $\Gamma^{\mathrm{enh}}(\mathcal{X},-)$ are always adjoints.

Definition

A stack is 1-*affine* if the adjunction

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A stack $\mathcal{X}$ is *(weakly)* 0-*affine* if $\mathrm{QCoh}(\mathcal{X}) \simeq \mathrm{Mod}_{\Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}})}$.

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Idea

A stack $\mathcal{X}$ is *(weakly)* 0-*affine* if $\mathrm{QCoh}(\mathcal{X}) \simeq \mathrm{Mod}_{\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})}$. Considering the tautological "categorical structure sheaf"

$$\mathbb{Q}\mathscr{C}\mathrm{oh}_{/\mathcal{X}} \colon \{\mathrm{Spec}(R) \to \mathcal{X}\} \mapsto \mathrm{Mod}_R,$$

we can interpret 1-affineness as a categorification of 0-affineness.

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3. Formal completions of quasi-compact and quasi-separated algebraic spaces along closed subsets with quasi-compact complement are 1-affine.
4. Ind-schemes are generally not 1-affine – e.g.,

$$\mathbb{A}^{\infty} := \varinjlim_{n} \mathbb{A}^{n}$$

is not 1-affine.

Sheaves of categories over Betti stacks

Let X be a space.

Fact

The 2-category $\mathrm{ShvCat}(X_{\mathrm{B}})$ *is naturally equivalent to the 2-category*

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of $\Bbbk$*-linear categorical local systems over* X*.*

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This is true because ShvCat sends colimits of prestacks to limits of presentable categories, and it is a sheaf for étale topology. So both $\mathrm{ShvCat}(X_{\mathrm{B}})$ and $\mathrm{LocSysCat}(X;\Bbbk)$ are equivalent to

$$\lim_{x\to X} \mathrm{Mod}_{\mathrm{Mod}_\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}.$$

The categorified monodromy equivalence

Theorem ([PPS24])

If X is a pointed simply connected topological space, there exists an equivalence of 2-categories

$$\mathrm{LocSysCat}(X;\Bbbk) \simeq \mathrm{LMod}_{\mathrm{LMod}_{\mathrm{C}_\bullet(\Omega^2_* X;\Bbbk)}}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}.$$

Sketch of the proof

1. Recall that, if $\mathscr{C}$ is cocomplete, $\mathrm{LocSys}(X;\mathscr{C}) \simeq \mathrm{LMod}_{\Omega_* X}(\mathscr{C})$. This applies to the case of $\mathscr{C} = \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ as well.

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2. One proves that for any presentably monoidal category $\mathscr{C}$ there exists an equivalence of 2-categories

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3. If X is simply connected then $\Omega_* X$ is connected, and one proves that the equivalence

$$\mathrm{LocSys}(\Omega_* X;\mathscr{C}) \simeq \mathrm{LMod}_{\Omega^2_* X}(\mathscr{C})$$

intertwines the Day convolution on the left hand side and the natural relative ($\mathbb{E}_1$-monoidal) tensor product over the $\mathbb{E}_2$-algebra $\Omega^2_* X$.

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4. Concatenating the equivalences thus obtained we deduce our claim.

Corollary: topological group actions

Corollary ([Tel14])

A topological group action of a connected topological group G on a presentable $\Bbbk$-linear category $\mathscr{C}$ is equivalent to the datum of an $\mathbb{E}_2$-algebra morphism

$$\mathrm{C}_\bullet(\Omega_* G;\Bbbk) \longrightarrow \mathrm{HH}^\bullet(\mathscr{C}) =: \mathrm{Ext}^\bullet(\mathrm{id}_{\mathscr{C}},\mathrm{id}_{\mathscr{C}}).$$

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A connected topological group G has the homotopy type of the based loop space $\Omega_* X$ where $X := \mathbf{B}G$. By the previous theorem, a G-action on $\mathscr{C}$ is equivalent to a left $\mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* G;\Bbbk)}$-module structure on $\mathscr{C}$.

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This is equivalent to a monoidal functor $\mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* G;\Bbbk)} \to \underline{\mathrm{Fun}}^{\mathrm{L}}(\mathscr{C}, \mathscr{C})$.
Finally, this is equivalent to an $\mathbb{E}_2$-algebra morphism $\mathrm{C}_\bullet(\Omega_* G;\Bbbk) \to \mathrm{HH}^\bullet(\mathscr{C})$. □

1-affineness for Betti stacks

Let X be a topological space. The 1-affineness problem for its Betti stack turns naturally into the following question: when do $\mathbb{k}$-linear categorical local systems over X coincide with modules over the category of $\mathbb{k}$-linear local systems?

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Proposition ([PPS24])

For every topological space X, the functor

$$\mathrm{Loc}_{X_{\mathrm{B}}} : \mathrm{Mod}_{\mathrm{LocSys}(X;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \longrightarrow \mathrm{LocSysCat}(X;\Bbbk)$$

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Proof (Slogan).

This is just a consequence of Barr-Beck-Lurie's monadicity theorem. □

1-affine topological stacks

Theorem

Betti stacks of 1*-truncated topological spaces are* 1*-affine.*

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Sketch of proof.

If X is 1-truncated, then $\Omega_* X$ is homotopy equivalent to a discrete group G. By a result of [Gai15], if $\mathscr{G}$ is a group prestack for which $\mathrm{Loc}_{\mathscr{G}}$ is fully faithful then $\mathbf{B}\mathscr{G}$ is 1-affine if and only if the global sections functor $\Gamma(\mathscr{G}, -)\colon \mathrm{QCoh}(\mathscr{G}) \to \mathrm{Mod}_{\Bbbk}$ is monadic.

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Applying this machinery to the group stack $\mathscr{G} := (\Omega_* X)_{\mathrm{B}} \simeq \Omega_*(X_{\mathrm{B}})$ we obtain that taking global sections

$$\Gamma(\Omega_* X, -) \simeq \prod_{g\in G} : \mathrm{LocSys}(\Omega_* X; \Bbbk) \simeq \prod_{g \in G} \mathrm{Mod}_{\Bbbk} \longrightarrow \mathrm{Mod}_{\Bbbk}$$

is indeed a monadic operation. □

Examples and counterexamples

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Conjecture (Conjecture / WIP)

Does every 1-affine Betti stack come from a 1-truncated topological space?

Sheaves of categories over coaffine stacks

One could also ask what happens when one chooses to consider sheaves of categories over the affinization of a Betti stack – namely, $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$.

Theorem ([PPS24])

If X is a pointed 2-connected topological space such that the algebra of $\Bbbk$-chains over its based loop space has finite $\Bbbk$-homology in each degree, then

$$\mathrm{aff}^*\colon \mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) \longrightarrow \mathrm{ShvCat}(X_{\mathrm{B}}) \simeq \mathrm{LocSysCat}(X;\Bbbk)$$

restricts to an equivalence between those sheaves of categories whose local sections over the point $\mathrm{Spec}(\Bbbk)$ *are dualizable in* $\mathrm{Mod}_{\mathrm{Mod}_\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$.

Remark

Since $\mathrm{LocSysCat}(X;\Bbbk) \simeq \mathrm{LMod}_{\mathrm{C}_\bullet(\Omega^2_*X;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$, this can be interpreted as a Koszul duality for categorified modules between $\mathrm{C}_\bullet(\Omega^2_*X;\Bbbk)$ and $\mathrm{C}^\bullet(X;\Bbbk)$.

Strategy of the proof

The proof again boils down to applying Barr-Beck-Lurie's (co)monadicity theorem to the commutative diagram of comonads

$$\begin{array}{ccc} \mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)) & \xrightarrow{\mathrm{aff}^*} & \mathrm{LocSysCat}(X;\Bbbk) \\ & {\scriptstyle \eta^*}\searrow \quad \swarrow{\scriptstyle \eta_{\mathrm{B}}^*} & \\ & \mathrm{Mod}_{\mathrm{Mod}_\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}. & \end{array}$$

where η^* and η_{B}^* are induced by the pointing $\eta\colon \{*\} \to X$.

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where η^* and η^*_{B} are induced by the pointing $\eta\colon \{*\} \to X$.
The only issue is that the diagram obtained from the one above by considering the right adjoints to η^* and η^*_{B}, in general, does not commute.

How presentability enters the picture

In order to make things work, the two essential ingredients are the following.

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1. Under our assumptions on X, we have

$$\mathrm{QCoh}(\mathrm{cSpec}(R \otimes_{\mathrm{C}^\bullet(X;\Bbbk)} \Bbbk)) \simeq \mathrm{LocSys}(\Omega_* X; \mathrm{Mod}_R)$$

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thanks to the Eilenberg-Moore spectral sequence.

2. If the category $\mathscr{C}$ of local sections over $\mathrm{Spec}(\Bbbk)$ is dualizable, then

$$\lim_{\mathrm{Spec}(R) \to \mathrm{cSpec}(R \otimes_{\mathrm{C}^\bullet(X;\Bbbk)} \Bbbk)} \mathscr{C} \otimes \mathrm{Mod}_R \simeq \mathscr{C} \otimes \mathrm{QCoh}(\mathrm{cSpec}(R \otimes_{\mathrm{C}^\bullet(X;\Bbbk)} \Bbbk))$$

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   is an equivalence.

Together, these facts imply our claim.

Failure of the equivalence

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Question

Is the above functor an equivalence if X is sufficiently connected or finite?

The category of n-categories

One can generalize the picture described up to this point to the n-categorical level.

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Definition

For $n \geqslant 2$, the category of (not necessarily small) n-categories is defined as

$$\widehat{\mathrm{Cat}}_{(\infty,n)} := \mathrm{Mod}_{\mathrm{Cat}_{(\infty,n-1)}} \widehat{\mathrm{Cat}}_{(\infty,1)}.$$

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Remark

These are actually categories; but considering the natural enrichment over themselves provided by the closed Cartesian monoidal structure, every $\widehat{\mathrm{Cat}}_{(\infty,n)}$ naturally upgrades to an $(n+1)$-category.

Presentable n-categories

In the setting of n-categories, the notion of presentability can be generalized as follows.

Definition ([Ste21])

Let $n \geqslant 2$, and let $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,n)}$ be the category of cocomplete n-categories. The category of presentable n-categories $\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ is the category of κ_0-compact objects

$$\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} := \mathrm{Mod}_{\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)}}\left(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,n)}\right)^{\kappa_0} \subseteq \mathrm{Mod}_{\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)}}\left(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,n)}\right),$$

where κ_0 is the smallest large cardinal for our theory.

Presentable n-categories II

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2. For every $n \geqslant 2$, the category $\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ is an $(n+1)$-category admitting all colimits (and a sufficient amount of limits) and equipped with a symmetric monoidal structure compatible with colimits in each variable.
3. One can take $\Bbbk$-linear coefficients by considering

$$\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} := \mathrm{Mod}_{\mathrm{Mod}_{\Bbbk}}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$$

and then defining

$$\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} := \mathrm{Mod}_{\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)}}\left(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,n)}\right)^{\kappa_0}.$$

Iterated modules

For every $n \geqslant 0$ and every $\mathbb{E}_{n+1}$-algebra A, we can produce a $\Bbbk$-linear presentable $(n+1)$-category of iterated left A-modules as follows.

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2. For $n \geqslant 1$, the category LMod_A^{n-1} is a presentably monoidal n-category. So we can consider the $(n+1)$-category

$$\mathrm{LMod}_A^n := \mathrm{LMod}_{\mathrm{LMod}_A^{n-1}}\left(\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)}\right).$$

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Remark

For $n \geqslant 1$, $\mathrm{Mod}_\Bbbk^n \simeq \mathrm{Lin}_\Bbbk \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$.

Higher categorical local systems

For any space, it is hence natural to consider the $(n+1)$-category

$$\mathrm{LocSysCat}^n(X;\Bbbk) := \mathrm{Fun}^n\left(\Pi_\infty(X),\, \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}\right)$$

of local systems of presentable n-categories.
We have this furtherly categorified monodromy equivalence.

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We have this furtherly categorified monodromy equivalence.

Theorem ([PPS24])

For every $n \geqslant 1$ and every n-connected topological space X, we have an equivalence of $(n+1)$-categories

$$\mathrm{LocSysCat}^n(X;\Bbbk) \simeq \mathrm{LMod}_{\mathrm{LMod}^n_{\mathrm{C}_\bullet\left(\Omega^{n+1}_* X;\Bbbk\right)}}\left(\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}\right).$$

Sketch of the proof

The strategy for the 2-categorical case works verbatim for every n, since the definition of $\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ is robust enough to retain all the key features of $\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ we needed in the first proof.
Then, one proves that the equivalence on the underlying categories intertwines the action of $\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ which provides the enrichment and, hence, their enhancement to $(n+1)$-categories.

Higher sheaves of categories and n-affineness

Definition ([Ste21])

Let $\mathcal{X}$ be a (pre)stack, and let $n \geqslant 1$.

1. The $(n+1)$-category of sheaves of n-categories is
$$\mathrm{ShvCat}^n(X) := \lim_{\substack{\mathrm{Spec}(R)\to\mathcal{X} \\ R\in\mathrm{CAlg}_{\Bbbk}^{\geqslant 0}}} \mathrm{Mod}_R^n$$
where the limit is computed in $\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n+1)}$.
2. We say that $\mathcal{X}$ is n-affine if the global sections functor $\Gamma(\mathcal{X},-)\colon \mathrm{ShvCat}^n(X) \to \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ is monadic.

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Remark

For $n=1$ we obtain exactly the notion of 1-affineness of [Gai15].

n-affine Betti stacks

Theorem ([PPS24])

For any space X and any $n \geqslant 2$, the Betti stack X_{B} is n-affine precisely if $(\Omega_* X)_{\mathrm{B}}$ is $(n-1)$-affine.
In particular, every n-truncated space is n-affine.

Remark

If the conjecture

$$X_{\mathrm{B}} \text{ 1-affine } \Leftrightarrow X \text{ 1-truncated}$$

holds, then the Betti stack is n-affine *if and only if* X is n-truncated.

WIP: Sheaves of n-categories on coaffine stacks

Theorem ([PPS24])

Given X an $(n+1)$-connected space with suitably finiteness assumptions, then the affinization map $\mathrm{aff}\colon X_{\mathrm{B}} \to \mathrm{cSpec}(\mathrm{C}^{\bullet}(X;\Bbbk))$ *induces a functor*

$$\mathrm{aff}^{*}\colon \mathrm{ShvCat}^{n}(\mathrm{cSpec}(\mathrm{C}^{\bullet}(X;\Bbbk))) \longrightarrow \mathrm{LocSysCat}^{n}(X;\Bbbk)$$

which is an equivalence on some suitable sub-$(n+1)$-categories on both sides.

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Remark

Again, this produces a duality between categorified modules over the $\mathbb{E}_{n+1}$-Koszul dual algebras $\mathrm{C}_\bullet(\Omega_*^{n+1}X;\Bbbk)$ and $\mathrm{C}^\bullet(X;\Bbbk)$, which generalizes the classical Koszul duality between modules over the Koszul dual algebras $\mathrm{C}_\bullet(\Omega_*X;\Bbbk)$ and $\mathrm{C}^\bullet(X;\Bbbk)$.

References

[Gai15] Dennis Gaitsgory. "Sheaves of categories and the notion of 1-affineness". In: *Stacks and categories in geometry, topology, and algebra*. Vol. 643. Contemp. Math. Amer. Math. Soc., Providence, RI, 2015, pp. 127–225. DOI: 10.1090/conm/643/12899. URL: https://doi.org/10.1090/conm/643/12899.

[Lur11] Jacob Lurie. *Derived Algebraic Geometry VIII: Quasi-Coherent Sheaves and Tannaka Duality Theorems*. 2011. URL: http://people.math.harvard.edu/~lurie/papers/DAG-VIII.pdf.

[PPS24] J. Pascaleff, E. Pavia, and N. Sibilla. "Higher categorical systems and Koszul duality". To appear. 2024.

[Ste21] Germán Stefanich. "Higher Quasicoherent Sheaves". PhD Thesis. UC Berkeley, 2021. URL: https://escholarship.org/uc/item/19h1f1tv.

[Tel14] Constantin Teleman. "Gauge theory and mirror symmetry". In: *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II*. Kyung Moon Sa, Seoul, 2014, pp. 1309–1332.

[Toë06] Bertrand Toën. "Champs affines". In: *Selecta Math. (N.S.)* 12.1 (2006), pp. 39–135. ISSN: 1022-1824,1420-9020. DOI: 10.1007/s00029-006-0019-z. URL: https://doi.org/10.1007/s00029-006-0019-z.