t-structures on ∞-categories

with an application to mixed graded complexes

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Original framework

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Definition ([DP61], [Ver96])

A *triangulated category* is an additive category $\mathscr{C}$ with a *translation functor* $[1]\colon \mathscr{C} \to \mathscr{C}$ and a class of *exact triangles*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\delta} X[1]$$

which is closed under isomorphism and satisfies the set of axioms TR1, TR2, TR3, TR4, TR5.

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Definition ([BBD82])

A *t-structure* on a triangulated category $\mathscr{C}$ is the datum of two full subcategories $\left(\mathscr{C}_{\geqslant 0},\ \mathscr{C}_{\leqslant 0}\right)$, that we call the categories of *connective* and *coconnective* objects respectively, such that

1. $\mathscr{C}_{\geqslant 1} := \mathscr{C}_{\geqslant 0}[1] \subseteq \mathscr{C}_{\geqslant 0}$ and $\mathscr{C}_{\leqslant -1} := \mathscr{C}_{\leqslant 0}[-1] \subseteq \mathscr{C}_{\leqslant 0}$.
2. For all X in $\mathscr{C}_{\geqslant 0}$ and all Y in $\mathscr{C}_{\leqslant 0}$, $\mathrm{Hom}_{\mathscr{C}}(X,\ Y[-1]) \cong 0$.
3. All objects X of $\mathscr{C}$ sit in an exact triangle $Y \to X \to Z \to Y[1]$ with Y lies in $\mathscr{C}_{\geqslant 0}$ and Z in $\mathscr{C}_{\leqslant -1}$.

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The *heart* of the t-structure is $\mathscr{C}^{\heartsuit} := \mathscr{C}_{\geqslant 0} \bigcap \mathscr{C}_{\leqslant 0}$.

Connective and coconnective covers

- For any integer n, the natural inclusions $\iota_{\geqslant n} : \mathscr{C}_{\geqslant n} \hookrightarrow \mathscr{C}$ and $\iota_{\leqslant n} : \mathscr{C}_{\leqslant n} \hookrightarrow \mathscr{C}$ admit, respectively, a right adjoint $\tau_{\geqslant n} : \mathscr{C} \to \mathscr{C}_{\geqslant n}$ and a left adjoint $\tau_{\leqslant n} : \mathscr{C} \to \mathscr{C}_{\leqslant n}$ called respectively the *n-connective* and *n-coconnective cover*.

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- One has for all couples $m \leqslant n$ natural transformations $\tau_{\leqslant n} \to \tau_{\leqslant m}$ and $\tau_{\geqslant m} \to \tau_{\geqslant n}$, such that
  1. $\tau_{\leqslant m} \xrightarrow{\simeq} \tau_{\leqslant m} \circ \tau_{\leqslant n}$.
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- In particular, one can define the functor $\mathrm{H}_0 := \tau_{\geqslant 0} \circ \tau_{\leqslant 0}\colon \mathscr{C} \to \mathscr{C}^{\heartsuit}$. By shifting, one has $\mathrm{H}_n := \tau_{\geqslant n} \circ \tau_{\leqslant n} \simeq \mathrm{H}_0 \circ [-n]\colon \mathscr{C} \to \mathscr{C}^{\heartsuit}$.

Examples of t-structures and their hearts

1. The derived category $\mathrm{h}\mathscr{D}(\mathscr{A})$ of an abelian category $\mathscr{A}$ admits a canonical t-structure.
   - $\mathrm{h}\mathscr{D}^{\geqslant 0}(\mathscr{A}) := \{X_\bullet \mid \mathrm{H}_n(X_\bullet) \cong 0 \text{ for all } n < 0\}$.
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Theorem ([BBD82, Theorem 1.3.6])

The heart $\mathscr{C}^\heartsuit$ is an admissible abelian subcategory of $\mathscr{C}$ which is stable under extensions, and the additive functor $\mathrm{H}_0 \colon \mathscr{C} \to \mathscr{C}^\heartsuit$ turns any exact triangle of $\mathscr{C}$ into a long exact sequence of $\mathscr{C}^\heartsuit$.

Structural properties of t-structures

Definition ([BBD82])

Let $\mathscr{C}$ be a triangulated category with t-structure.

- The t-structure is *bounded below* (resp. *above*) if the inclusion $\mathscr{C}^- := \bigcup \mathscr{C}_{\geqslant n} \subseteq \mathscr{C}$ (resp. $\mathscr{C}^+ := \bigcup \mathscr{C}_{\leqslant n} \subseteq \mathscr{C}$) is an equivalence.

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Some properties of $\mathscr{C}$ and pathological behaviors of t-structures can be often expressed in terms of these properties.

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The problem lies in the fact that in triangulated categories many important constructions are unique up to a *non-unique* isomorphism.

Motivation for ∞-categories

In order to solve these problems, one needs to keep track of all the *homotopy coherences*. While this appears somewhat complicated in the 1-categorical world, one achieves this naturally in the ∞-categorical world.

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Nowadays, the canon introduction to this theory is [Lur09].

"Definition" of ∞-categories

Definition (Slogan)

An ∞-category is a category with k-morphisms for all $k \geqslant 0$ defined in the following way.

- For $k = 0$, 0-morphisms are the *objects*.
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Alternatively, one can imagine ∞-categories to be categories enriched in the homotopy category $\mathrm{hTop} := \mathrm{Top}\left[\mathscr{W}^{-1}\right]$.

Relationship with the 1-categorical framework

- The homotopy type of a topological space X is encoded in its *fundamental ∞-groupoid*, i.e., an ∞-category $\Pi_\infty(X)$ whose k-morphisms are the set of homotopical classes of k-dimensional paths of X.

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- Given a 1-category $\mathscr{C}$ with a class of weak equivalences $\mathscr{W}$, *Dwyer-Kan localization* produces an ∞-category $\mathrm{L}^{\mathrm{H}}(\mathscr{C})$ such that $\mathrm{h}\left(\mathrm{L}^{\mathrm{H}}(\mathscr{C})\right)$ agrees with the homotopy category $\mathscr{C}\left[\mathscr{W}^{-1}\right]$.

∞-categorical constructions

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1-categories	∞-categories
Set	$\mathcal{S}$
Hom-set	Mapping space
Isomorphism	Homotopy equivalence
Limits and colimits	Homotopy limits and colimits
Adjunction	∞-adjunction
Presheaves with Set-values	Presheaves with $\mathcal{S}$-values
Yoneda embedding	∞-Yoneda embedding
Grothendieck topos	∞-topos

Algebra in ∞-categories

Given a pointed ∞-category $\mathscr{C}$ with zero object 0, a square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

is a *fiber* (resp. *cofiber*) *sequence* if it is a pullback (resp. pushout) diagram.

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A pointed ∞-category $\mathscr{C}$ is *stable* if all morphisms admit a fiber and a cofiber, and a triangle is a fiber sequence if and only if it is a cofiber sequence.

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The *looping* (resp. *suspension*) of an object X is $X[-1] := \mathrm{fib}(0 \to X)$ (resp. $X[1] := \mathrm{cofib}(X \to 0)$).

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- *An ∞-functor $F \colon \mathscr{C} \to \mathscr{D}$ between stable ∞-categories is left exact if and only if it is right exact.*
- *The homotopy category of a stable ∞-category $\mathscr{C}$ is always triangulated, with exact triangles given by the homotopy classes of fiber/cofiber sequences.*

Stable enhancements of well known triangulated categories

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2. In particular, for every ordinary ring R there exists stable ∞-categories of left modules LMod_R and right modules RMod_R. If it is commutative, they are both equivalent to the other and are denoted by Mod_R.
3. The stabilization $\mathrm{Sp} := \mathrm{Sp}(\mathscr{S})$ of the ∞-category of homotopy types is the *stable ∞-category of spectra*, and its homotopy category is the stable homotopy category hSp.

Definition of t-structures on stable ∞-categories

Definition (Generalization of t-structures to stable ∞-categories)

A t-structure on a stable ∞-category $\mathscr{C}$ is the datum of two full sub-∞-categories $\left(\mathscr{C}_{\geqslant 0},\ \mathscr{C}_{\leqslant 0}\right)$ such that

1. For all X in $\mathscr{C}_{\geqslant 0}$ and all Y in $\mathscr{C}_{\leqslant 0}$, $\mathrm{Map}_{\mathscr{C}}(X,\ Y[-1]) \simeq \{*\}$.
2. $\mathscr{C}_{\geqslant 1} := \mathscr{C}_{\geqslant 0}[1] \subseteq \mathscr{C}_{\geqslant 0}$ and $\mathscr{C}_{\leqslant -1} := \mathscr{C}_{\leqslant 0}[-1] \subseteq \mathscr{C}_{\leqslant 0}$.
3. For all objects X of $\mathscr{C}$ there exists a fiber/cofiber sequence $Y \to X \to Z \to Y[1]$ with Y in $\mathscr{C}_{\geqslant 0}$ and Z in $\mathscr{C}_{\leqslant -1}$.

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Definition (t-structure on a stable ∞-category as in [Lur17])

A t-structure on a stable ∞-category is a classical t-structure on its homotopy category.

Filtered objects in stable ∞-categories

Definition ([Ill71], [Bei87])

The ∞-category of filtered $\Bbbk$-modules $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$ for a commutative ring $\Bbbk$ is the ∞-category of ∞-functors $\mathrm{Fun}\big(\mathbb{Z}_{\geqslant}, \mathrm{Mod}_{\Bbbk}\big)$, i.e. the ∞-category of sequences of $\Bbbk$-modules

$$M_\bullet \colon \ldots \to M_{n+1} \to M_n \to M_{n-1} \to \ldots.$$

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The ∞-category of filtered $\Bbbk$-modules $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$ for a commutative ring $\Bbbk$ is the ∞-category of ∞-functors $\mathrm{Fun}\big(\mathbb{Z}_{\geqslant}, \mathrm{Mod}_{\Bbbk}\big)$, i.e. the ∞-category of sequences of $\Bbbk$-modules

$$M_{\bullet}\colon \ldots \to M_{n+1} \to M_{n} \to M_{n-1} \to \ldots.$$

Given a filtered $\Bbbk$-module $M_{\bullet}$,

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3. Its *associated graded complex* is $\mathrm{Gr}_\bullet X_\bullet := \left\{\mathrm{Gr}_q M_\bullet := \operatorname{cofib}\left(M_{q+1} \to M_q\right)\right\}_{q\in\mathbb{Z}}$.

Beilinson t-structure in the stable setting

Theorem ([Bei87], [BMS19])

The ∞-category of filtered $\Bbbk$-modules $\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$ admits a t-structure described in the following way.

- $\left(\mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}\right)_{\geqslant 0} := \left\{M_\bullet \mid \mathrm{H}_n\left(\mathrm{Gr}_q M_\bullet\right) \cong 0 \textit{ for all } n < -q\right\}.$
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There exist filtered $\Bbbk$-modules $M_\bullet$ such that $\mathrm{H}_n^{\mathrm{fil}}(M_\bullet) := \tau_{\geqslant n} \circ \tau_{\leqslant n}(M_\bullet)$ is trivial for any n, without being trivial themselves: the Beilinson t-structure is *not left separated* or *left complete*.

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Mixed graded complexes

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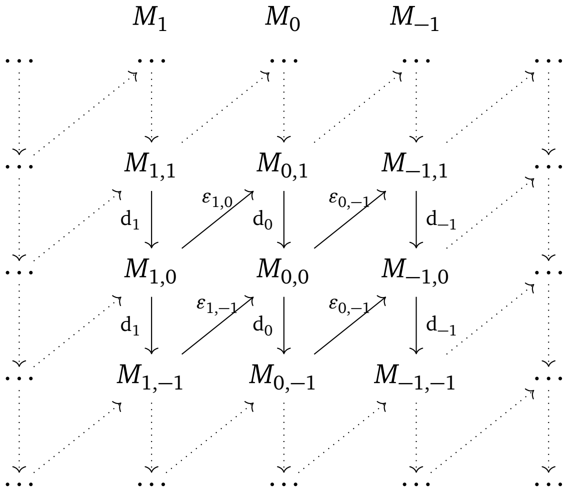
Definition ([PTVV13])

Let $\Bbbk$ be a Noetherian commutative $\mathbb{Q}$-algebra. A *mixed graded complex of* $\Bbbk$*-modules* is a graded complex $M_\bullet := \{M_q\}_{q\in\mathbb{Z}}$ with a collection of morphisms $\{\varepsilon_q : M_q \to M_{q-1}[-1]\}$ such that $\varepsilon_{q-1}[-1] \circ \varepsilon_q = 0$.
By Dwyer-Kan localizing, one gets the ∞*-category of mixed graded* $\Bbbk$*-modules* $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$.

Visualizing mixed graded complexes

$M_\bullet$:



Mixed graded complexes as complete filtered complexes

Theorem ([TV20], [P20XX])

The stable ∞-category $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}}$ admits a t-structure described in the following way.

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Moreover, there exists a fully faithful t-exact embedding $\varepsilon\text{-}\mathrm{Mod}_{\Bbbk}^{\mathrm{gr}} \hookrightarrow \mathrm{Mod}_{\Bbbk}^{\mathrm{fil}}$ which induces an equivalence on those complexes with complete filtrations.

A. A. Beilinson. "On the derived category of perverse sheaves". In: *K-theory, arithmetic and geometry (Moscow, 1984–1986)*. Vol. 1289. Lecture Notes in Math. Springer, Berlin, 1987, pp. 27–41. DOI: 10.1007/BFb0078365. URL: https://doi-org.pros.lib.unimi.it:2050/10.1007/BFb0078365.

A. A. Beilinson, J. Bernstein, and P. Deligne. "Faisceaux pervers". In: *Analysis and topology on singular spaces, I (Luminy, 1981)*. Vol. 100. Astérisque. Soc. Math. France, Paris, 1982, pp. 5–171.

Bhargav Bhatt, Matthew Morrow, and Peter Scholze. "Topological Hochschild homology and integral p-adic Hodge theory". In: *Publ. Math. Inst. Hautes Études Sci.* 129 (2019), pp. 199–310. ISSN: 0073-8301. DOI: 10.1007/s10240-019-00106-9. URL: https://doi-org.pros.lib.unimi.it:2050/10.1007/s10240-019-00106-9.

J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces*. Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, Berlin-New York, 1973, pp. x+257.

Albrecht Dold and Dieter Puppe. "Homologie nicht-additiver Funktoren. Anwendungen". In: *Ann. Inst. Fourier (Grenoble)* 11 (1961), pp. 201–312.

ISSN: 0373-0956. URL: http://www.numdam.org/item?id=AIF_1961__11__201_0.

Luc Illusie. *Complexe cotangent et déformations. I.* Lecture Notes in Mathematics, Vol. 239. Springer-Verlag, Berlin-New York, 1971, pp. xv+355.

Jacob Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0; 0-691-14049-9. DOI: 10.1515/9781400830558. URL: https://doi.org/10.1515/9781400830558.

Jacob Lurie. *Higher Algebra*. 2017. URL: http://www.math.harvard.edu/~lurie/papers/HA.pdf.

Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi. "Shifted symplectic structures". In: *Publ. Math. Inst. Hautes Études Sci.* 117 (2013), pp. 271–328. ISSN: 0073-8301. DOI: 10.1007/s10240-013-0054-1. URL: https://doi.org/10.1007/s10240-013-0054-1.

Daniel G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967, iv+156 pp. (not consecutively paged).

Charles Rezk. “A model for the homotopy theory of homotopy theory”. In: *Trans. Amer. Math. Soc.* 353.3 (2001), pp. 973–1007. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-00-02653-2. URL: https://doi-org.pros1.lib.unimi.it/10.1090/S0002-9947-00-02653-2.

Bertrand Toën and Gabriele Vezzosi. *Algebraic foliations and derived geometry II: the Grothendieck-Riemann-Roch theorem*. 2020. arXiv: 2007.09251 [math.AG].

Jean-Louis Verdier. “Des catégories dérivées des catégories abéliennes”. In: *Astérisque* 239 (1996). With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis, xii+253 pp. (1997). ISSN: 0303-1179.