An introduction to derived algebraic geometry: Calabi-Yau structures and symplectic stacks

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Why was derived algebraic geometry introduced?

Although the proper mathematical foundations of derived algebraic geometry date back only to the beginning of the new century, ideas which would be later recognized as *derived* have arised naturally since the late Fifties. In order to tackle problems concerning classical algebraic geometry, mathematicians started employing techniques from higher category theory and algebraic topology.

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Examples (Applications of derived geometry)

- Intersection theory.
- Representability of moduli problems.
- Quotient of schemes by non-free action of algebraic groups.

Why is derived algebraic geometry interesting nowadays?

With the development of the appropriate technical machinery and theoretical framework in the last twenty years, derived algebraic geometry is nowadays a valuable tool which is used to formulate, study and understand problems coming not only from classical algebraic geometry, but also from algebraic topology, symplectic and Poisson geometry, algebraic number theory and theoretical physics.

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Examples

- ([AG15]) Geometric Langlands correspondence.
- ([Bei12]) p-adic cohomology.
- ([Lur09a]) Topological modular forms.
- ([KS01]) Homological mirror symmetry.

First steps of derived algebraic geometry

The mathematical result which, *a posteriori*, is considered to mark the birth of derived algebraic geometry (or at least of some of its key features) is *Serre's intersection multiplicity formula* (1958).

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Theorem ([Ser65])

Let $(X, \mathscr{O}_X)$ be a smooth algebraic variety, V and W two subvarities with complementary dimensions corresponding respectively to two sheaf of ideals $\mathscr{I}_V$ and $\mathscr{I}_W$ in $\mathscr{O}_X$. Given $x \in V \cap W$, let $\mathscr{O}_{X,x}$ be the stalk at x.
The multiplicity of $V \cap W$ at the point x is:

$$\sum_{i=0}^{+\infty} (-1)^i \ell_{\mathscr{O}_{X,x}} \left(\mathrm{Tor}_i^{\mathscr{O}_{X,x}} \left(\mathscr{O}_{X,x}/\mathscr{I}_V,\ \mathscr{O}_{X,x}/\mathscr{I}_W \right) \right)$$

First steps of derived algebraic geometry

Serre's intersection formula has to take into account *all* the Tor-modules in order to count non-transverse intersections with the correct multiplicity, even if the scheme-theoretic intersection of the two subvarieties is governed only by the ordinary tensor product of rings $\mathscr{O}_{V,x} \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{W,x}$.

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A step forward towards derived algebraic geometry is due to André-Quillen and Grothendieck-Illusie, who (independently) came up with the notion of *cotangent complex* in order to study the deformation theory of commutative rings and schemes.

Cotangent complex formalism

Definition ([Qui70], [Ill71])

The cotangent complex of a morphism $f\colon A \longrightarrow B$ of commutative rings is the differential graded B-module:

$$\mathbb{L}_{B/A} := \mathrm{C}_\bullet\left(\Omega_{A^\bullet/A} \otimes_{A^\bullet} B\right)$$

where $A^\bullet \longrightarrow B$ is any simplicial resolution of B by free A-algebras and $\Omega_{-/A}$ is the Kähler differential functor.

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- Illusie's definition depends on a specific choice of a simplicial resolution.
- Quillen and André proved that the cotangent complex is independent from the choice of the simplicial resolution.

Model categories

Definition ([Hov99])

A model structure on a category $\mathcal{C}$ provided with all small limits and colimits is the choice of three classes of morphisms of $\mathcal{C}$ (the cofibrations $\mathscr{C}$, the fibrations $\mathscr{F}$ and the weak equivalences $\mathscr{W}$), satisfying:

- Retracts of morphisms in a distinguished class belong to the same class of morphisms;
- If two between f, g and $f \circ g$ are weak equivalences, so is the third;
- Fibrations have the right lifting property with respect to trivial cofibrations, and the dual holds.
- Every morphism of $\mathcal{C}$ can be factorized both as a cofibration followed by a trivial fibration, and as a trivial cofibration followed by a fibration.

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- The category Set_Δ of simplicial sets, with $\mathscr{C}$ given by injective morphisms and $\mathscr{W}$ given by morphisms whose geometric realization is a homotopy equivalence, is a model category.

Basic facts of model categories

Given a model category $\mathcal{C}$, we can always talk about:

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Remark

For any category $\mathcal{C}$ and for any class of morphisms $\mathscr{W}$ in $\mathcal{C}$ we can consider the localization $\mathcal{C}[\mathscr{W}^{-1}]$ of $\mathcal{C}$ at $\mathscr{W}$, sending each morphism of $\mathscr{W}$ to an isomorphism, but its description is in general quite cumbersome.

Homotopy category of a model category

However, for model categories, the localization $\mathcal{C}[\mathscr{W}^{-1}]$ is much more easily understandable.

Theorem (Structure of the homotopy category, [Hov99])

Let $\mathcal{C}$ be a model category. Let $\mathcal{C}^{\mathrm{cf}}$ be the category consisting of all the objects X of $\mathcal{C}$ which are both fibrant and cofibrant, and given two cofibrant-and-fibrant objects X and Y consider the quotient of the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of the morphisms between X and Y in $\mathcal{C}$ modulo the equivalence relation of homotopy equivalence.
Then $\mathrm{Ho}\,\mathcal{C}$ is equivalent to the category $\mathcal{C}^{\mathrm{cf}}$ with morphisms given by the construction described above, and it is equivalent to the localization $\mathcal{C}[\mathscr{W}^{-1}]$.

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- Model categories provide a presentation and a model to work effectively in their homotopy category.
- Model categories provide the correct sub-classes of objects and morphisms which better approximate the behaviour of all objects and morphisms in the homotopy setting.
- Ultimately, model categories are the first environment in which one chooses to relax the identity relation: two things are indistinguishable whenever there is a homotopy between them.

Relationship between homotopical algebra and derived algebraic geometry

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- Given a left Quillen functor between two model categories $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ (that is, a functor which preserves cofibrations and trivial cofibrations) one can *derive* it, i.e. extend it to the homotopy category $\mathbb{L}F\colon \operatorname{Ho}\mathcal{C} \longrightarrow \operatorname{Ho}\mathcal{D}$. $\mathbb{L}F$ is the *left derived functor* of F.

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- With these notations, the Tor-functor in Serre's formula is the left derived functor of the tensor product endofunctor of $\mathrm{C}_\bullet\,\mathrm{Mod}_R$.
- With these notations, the cotangent complex $\mathbb{L}_{B/A}$ is the image of B via the left derived functor of the Kähler differential functor $\Omega_{B/A}\colon\ \left(\mathrm{CRing}_{/B}\right)_\Delta \longrightarrow (\mathrm{Mod}_B)_\Delta$.

Drawbacks of model categories

- Even fundamental operations between fibrant-and-cofibrant objects in the model category can produce an object which is not fibrant-and-cofibrant anymore.

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- We always need to replace all the objects by suitable fibrant-and-cofibrant models.
- In general, we have always to check that what we do at the level of model categories respects the underlying structure of the homotopy category.
- Many constructions in the homotopy category cannot be carried out in a functorial fashion or do not have desired properties, because the homotopy category *forgets* the homotopies between objects and morphisms.

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Conceptually, they are categories with objects, (1-)morphisms between objects, 2-morphisms between morphisms, and in general k-morphisms between $(k-1)$-morphisms for all k in $\mathbb{N}$, and moreover any k-morphism is an equivalence when $k \geq 2$.

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Formally, they are categories *enriched* in topological spaces, that is we have not a set of morphisms between two objects, but a whole topological space (that we want to consider only up to homotopy). The 1-morphisms are points, the 2-morphisms are paths between points, the 3-morphisms are homotopies between paths, and so on.

Models for ∞-categories

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Definition ([Lur09b])

A quasi-category (or weak Kan complex) is a simplicial set X such that for all $1 \leq i \leq n-1$ and all diagrams of the form:

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Remark

- If $\mathcal{C}$ is a (small) category, one can build a simplicial set $\mathrm{N}\,\mathcal{C}$ (the *nerve of* $\mathcal{C}$) having as vertices the set of objects, as edges the set of morphisms, and for all $n > 1$ the n-simplices are given by composition of compatible n morphisms. Then $\mathrm{N}\,\mathcal{C}$ satisfies the filling property for all $1 \leq i \leq n-1$, and the filler is actually unique.

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- If X is a topological space, the simplicial set $\mathrm{Sing}(X)_\bullet = \mathrm{Hom}_{\mathrm{Top}}(\Delta^\bullet, X)$ satisfies the filling property for all $0 \leq i \leq n$.

∞-categories, topological spaces and categories

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The model of quasi-categories makes it quite easy to talk about ∞-functors, composition of ∞-functors, and the ∞-category of functors between ∞-categories.

Technical issues

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- It is illogical to require Hom-spaces of objects with universal properties to consist only of a single morphism: the correct ∞-categorical request is their Hom-spaces to be *contractible*.
- Diagrams must not commute strictly, but have to be commutative *up to coherent homotopy*.

A silver lining

Still, it is possible to work effectively in the ∞-categorical world. Not only ∞-categories compactify heavily the notations and the language of homotopical algebra, but many constructions of the (1-)categorical world generalize to ∞-categories.

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- One can talk about Grothendieck topologies and ∞-toposes.

Presentable ∞-categories

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- There exists another (Joyal) model structure on Set_Δ whose simplicial localization yields the ∞-category Cat_∞ of (small) ∞-categories.

Homotopical algebra: some analogies

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Definition

A commutative differential graded algebra (cdga for short) is a graded algebra $(A, \cdot)$ with a differential $\mathrm{d}\colon A \longrightarrow A[-1]$ such that $\mathrm{d}(a \cdot b) = \mathrm{d}(a) \cdot b + (-1)^{\deg(a)} a \cdot \mathrm{d}(b)$, $\mathrm{d} \circ \mathrm{d} = 0$ and $a \cdot b = (-1)^{\deg(a)\deg(b)} b \cdot a)$.

Definition

- A *derived scheme* X is a pair $(X, \mathscr{O}_X)$, where X is a topological space and $\mathscr{O}_X$ is an ∞-sheaf of derived commutative algebras such that $(X, \mathrm{H}_0(\mathscr{O}_X))$ is a scheme and for all $i > 0$ $\mathrm{H}_i(\mathscr{O}_X)$ is a quasi-coherent $\mathrm{H}_0(\mathscr{O}_X)$-module.

Derived schemes

Definition

- A *derived scheme* X is a pair $(X, \mathscr{O}_X)$, where X is a topological space and $\mathscr{O}_X$ is an ∞-sheaf of derived commutative algebras such that $(X, \mathrm{H}_0(\mathscr{O}_X))$ is a scheme and for all $i > 0$ $\mathrm{H}_i(\mathscr{O}_X)$ is a quasi-coherent $\mathrm{H}_0(\mathscr{O}_X)$-module.
- A morphism of derived schemes $(f, f^\sharp) : (X, \mathscr{O}_X) \longrightarrow (Y, \mathscr{O}_Y)$ is a morphism of the underlying topological spaces $f : X \longrightarrow Y$ together with a morphism of ∞-sheaves of derived commutative algebras $f^\sharp : \mathscr{O}_Y \longrightarrow f_*\mathscr{O}_X$.

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Derived schemes are gathered in the ∞-category dSch.

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- As in the classical case, one can obtain from a derived commutative algebra A a derived scheme $\mathrm{Spec}(A)$ having A as (derived) structure sheaf. Derived schemes which are (equivalent to ones) obtained in this way are called *affine* derived schemes, and make up a full sub-∞-category dAff of dSch.

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- One has an equivalence of ∞-categories $\mathrm{dAff} \simeq (\mathrm{CAlg})^{\mathrm{op}}$.
- Given a morphism of derived schemes $f\colon X \longrightarrow Y$ we say that f has a property $\mathcal{P}$ (e.g. it is proper, smooth, quasi-compact, étale, Zariski open immersion...) if the induced morphism of schemes $\tilde{f}\colon (X, \mathrm{H}_0(\mathscr{O}_X)) \longrightarrow (Y, \mathrm{H}_0(\mathscr{O}_Y))$ has $\mathcal{P}$ and for all $i > 0$ one has $\mathrm{H}_i(\mathscr{O}_Y) \otimes_{\mathrm{H}_0(\mathscr{O}_Y)} \mathrm{H}_0(\mathscr{O}_X) \cong \mathrm{H}_i(\mathscr{O}_X)$.

Derived stacks

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A derived stack is an ∞-functor $F \colon \mathrm{dAff}^{\mathrm{op}} \longrightarrow \mathcal{S}$ such that for all étale hypercovers $\{\mathrm{Spec}(A_i) \longrightarrow \mathrm{Spec}(B)\}$ of derived affine schemes the family of morphisms of spaces $\{F(B) \longrightarrow F(A_i)\}$ exhibits $F(B)$ as a homotopy limit for $F(A_i)$.

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Derived stacks are also endowed with a structure sheaf of derived commutative rings $\mathscr{O}_F$.

Quasi-coherent and perfect modules on derived stacks

As in the classical setting, given a derived stack F one has two important ∞-categories associated to it.

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These ∞-categories are moreover *stable*. In particular, their homotopy categories are Abelian categories, and if F corresponds to the derived stack $\mathrm{Map}(-, X)$ for a derived scheme X, then one has ([Lur17, Section 7.1.1]):

- $\mathrm{Ho}\,\mathrm{QCoh}(X) \simeq \mathrm{QCoh}\left(\mathrm{H}_0\left(\mathscr{O}_X\right)\right)$;
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Derived stacks and their stable ∞-categories of perfect modules are objects of fundamental interest in the field of derived symplectic geometry.

Why derived symplectic geometry?

Derived techniques are used nowadays in the context of symplectic geometry in order to tackle Kontsevich's *homological mirror symmetry conjecture* (HMS).

Conjecture (HMS)

For any complex Calabi-Yau symplectic manifold X (A-model), there exists a complex Calabi-Yau algebraic variety $X^{\wedge}$ (B-model) such that the Fukaya stable ∞-category of X is equivalent to the derived stable ∞-category of coherent sheaves of $X^{\wedge}$.

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In this last section, we will develop the formalism of symplectic structures on derived stacks and the formalism of Calabi-Yau structures on stable ∞-categories.

The cotangent complex formalism

Given a derived commutative ring R defined over a field k of characteristic 0 with cotangent complex $\mathbb{L}_{R/k}$, we can consider the *de Rham algebra* of R $\mathrm{DR}(R) := \mathrm{Sym}_k\left(\mathbb{L}_{R/k}\right)$, which is a *mixed graded commutative algebra*:

$$\begin{array}{ccccc}
R & & 0 & & 0 \\
\uparrow & \overset{\mathrm{d}_{\mathrm{dR}}}{\searrow} & \uparrow & & \uparrow \\
0 & & \mathbb{L}_{R/k} & & 0 \\
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The idea is that de Rham differential is geometric in nature, and so it should be distinguished from the differential coming from the derived structure of R.

Shifted p-forms and closed p-forms

Definition ([PTVV13])

- The *space of n-shifted p-forms* $\mathcal{A}^{p,\mathrm{cl}}(R,\, n)$ on R is the underlying space of $\bigwedge^p \mathbb{L}_{R/k}[n]$.
- The *space of n-shifted closed p-forms* $\mathcal{A}^{p,\mathrm{cl}}(R,\, n)$ on R is the space $\mathrm{Map}_{\varepsilon\text{-}\mathrm{Mod}_k^{\mathrm{gr}}}(k(-p)[p-n],\, \mathrm{DR}(R))$.

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By descent, one can talk about n-shifted p-forms and n-shifted closed p-forms also for derived stacks with cotangent complex. When F is a stack coming from a discrete (i.e. classical) scheme X, then 0-shifted (closed) p-forms on F are just classical (closed) p-forms on X.

n-shifted symplectic forms

When $p = 2$ and the cotangent complex $\mathbb{L}_{F/k}$ is dualizable (with dual $\mathbb{T}_{F/k}$), an n-shifted 2-form corresponds to a morphism $\tilde{\omega}\colon \mathbb{T}_{F/k}[-n] \longrightarrow \mathbb{L}_{F/k}$. ω is said to be *non-degenerate* if $\tilde{\omega}$ is an equivalence.

Definition ([PTVV13])

The space of n-shifted symplectic forms on a derived stack F is the space $\mathrm{Symp}\,(F,\ n)$ of closed p-forms which are also non-degenerate.

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Example

If X is a smooth complex algebraic variety there are no n-shifted 2-forms for $n \neq 0$, and a closed 0-shifted 2-form is non-degenerate if and only the corresponding classical 2-form on X is non-degenerate in the classical sense.

Lagrangian correspondences

Derived stacks with given symplectic structure cannot be collected in some sub-∞-category of derived stacks, since morphisms of stacks do not respect the symplectic structure.

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Definition ([Hau18])

- An n-Lagrangian structure on a morphism $f\colon Z \longrightarrow (X,\, \omega_X)$ with n-symplectic target is the datum of an homotopy between $f^*\omega_X$ and 0, which is non-degenerate.
- The ∞-category of *n-shifted Lagrangian correspondences* is the ∞-category having as objects pairs $(X,\, \omega_X)$ consisting of a derived stack and an n-shifted symplectic structure on it, and as morphisms spans $(X,\, \omega_X) \longleftarrow (Z,\, \omega_Z) \longrightarrow (Y,\, \omega_Y)$ inducing a Lagrangian structure on $Z \longrightarrow (X \times Y,\, \omega_X - \omega_Y)$.

Modules on stable ∞-categories

Mirror symmetry applies to Calabi-Yau varieties and manifolds. In particular, we need to introduce their non-commutative analogue, i.e. *stable Calabi-Yau ∞-categories.*

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We assume some technical hypothesis on our stable ∞-categories from now on.

Definition

Let $\mathcal{C}$, $\mathcal{D}$ be stable k-linear ∞-categories.
A $(\mathcal{C}, \mathcal{D})$-bimodule is an exact functor $\mathcal{C} \otimes_k \mathcal{D}^{\mathrm{op}} \longrightarrow \mathrm{Mod}_k$.

Hochschild homology

Example

For any $\mathcal{C}$ stable k-linear ∞-category, we have the diagonal $(\mathcal{C},\ \mathcal{C})$-bimodule $\underline{\mathcal{C}}\colon \mathcal{C}\otimes_k \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Mod}_k$ given by the association $(x,\ y)\mapsto \mathrm{Map}_{\mathcal{C}}(x,\ y)$.

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Given two $(\mathcal{C},\ \mathcal{C})$-bimodules M and N, one can consider the tensor product $M \otimes_{\mathcal{C} \otimes_k \mathcal{C}^{\mathrm{op}}} N$, which is a (derived) k-module.

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Definition

The *Hochschild complex* of a stable k-linear idempotent complete ∞-category $\mathcal{C}$ is the k-module $\mathrm{HH}(\mathcal{C}) := \underline{\mathcal{C}} \otimes_{\mathcal{C} \otimes_k \mathcal{C}^{\mathrm{op}}} \underline{\mathcal{C}}$.

d-dimensional Calabi-Yau structures on stable k-linear ∞-categories

The Hochschild complex comes equipped with an action of $S^1 \simeq k \otimes_{k \otimes k} k$. In particular, we can consider the homotopy orbits and the homotopy fixed points of $\mathrm{HH}(\mathcal{C})$.

Definition

- The *cyclic complex* of $\mathcal{C}$ is $\mathrm{HC}(\mathcal{C}) := \mathrm{HH}(\mathcal{C})_{S^1}$.
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Definition ([BD19])

A *d-dimensional Calabi-Yau structure on a stable k-linear ∞-category* $\mathcal{C}$ is a morphism of k-modules $\eta\colon k[d] \longrightarrow \mathrm{HC}^-(\mathcal{C})$ which is non-degenerate (in a suitable sense).

Relation between derived Calabi-Yau structures and classical Calabi-Yau varieties

- Given a complex smooth and proper Kähler algebraic variety X of dimension d, we say that X is *Calabi-Yau* if there is a trivialization of the canonical line bundle $\omega_X = \bigwedge^d \Omega^1_{X/\mathbb{C}} \simeq \mathscr{O}_X$. This request is equivalent to asking the existence of a nowhere vanishing holomorphic (closed) d-form.

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- One has that a choice of a trivialization of ω_X induces a non-degenerate closed d-form in $\mathbb{L}_{X/k}$, hence a d-homology class in $\mathrm{HH}(X) := \mathscr{O}_X \otimes_{\mathscr{O}_X \otimes_{\mathbb{C}} \mathscr{O}_X} \mathscr{O}_X$ by the HKR theorem. Since $\mathrm{HH}(X) \simeq \mathrm{HH}(\mathrm{Perf}(X))$, this machinery induces a derived d-dimensional Calabi-Yau structure on the stable $\mathbb{C}$-linear ∞-category $\mathrm{Perf}(X)$.

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- On the converse, a d-dimensional Calabi-Yau structure on $\mathrm{Perf}(X)$ induces a choice of a trivialization of ω_X.

State of the art

$\mathrm{Stab}_\infty(k((q)))$

Mirror symmetry involves a suitable subring of the formal power series over a ring of characteristic 0

$\mathrm{dSt}_{k((q))}$

State of the art

Moduli of objects

$$\mathrm{Stab}_{\infty}(k((q)))$$

$$\mathcal{M}_{-}$$

$$\downarrow$$

$$\mathrm{dSt}_{k((q))}$$

State of the art

Variations of Hodge structures

$$\begin{array}{ccc} \mathrm{Stab}_{\infty}(k((q))) & & \\ \downarrow{\scriptstyle \mathcal{M}_{-}} & & \mathrm{VHS}_{k((q))} \\ \mathrm{dSt}_{k((q))} & & \end{array}$$

State of the art

Gromov-Witten invariants

$\mathrm{Stab}_\infty(k((q)))$

$\mathcal{M}_-$ ↓

$\mathrm{dSt}_{k((q))}$

$\mathrm{VHS}_{k((q))}$

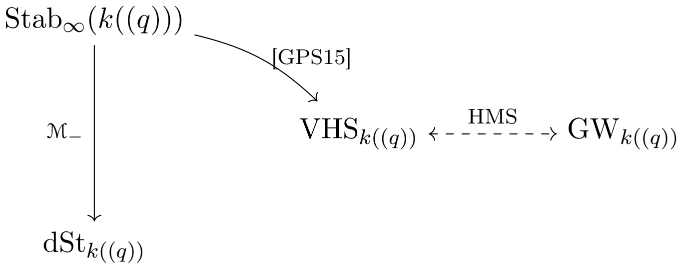
$\mathrm{GW}_{k((q))}$

State of the art

$$\mathrm{Stab}_\infty(k((q))) \xrightarrow{\ \mathcal{M}_-\ } \mathrm{dSt}_{k((q))}$$

$$\mathrm{VHS}_{k((q))} \overset{\mathrm{HMS}}{\dashleftarrow\dashrightarrow} \mathrm{GW}_{k((q))}$$

State of the art



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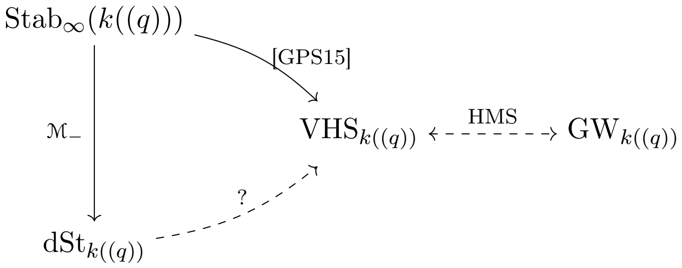

State of the art

$$\mathrm{VHS}_{k((q))} \overset{\mathrm{HMS}}{\longleftarrow\!\!-\!\!-\!\!\longrightarrow} \mathrm{GW}_{k((q))}$$

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$$\mathrm{VHS}_{k((q))} \overset{\mathrm{HMS}}{\longleftarrow\!\!-\!-\!-\!\longrightarrow} \mathrm{GW}_{k((q))}$$

$$\mathrm{Lag}^{2-d}_{k((q))}$$

State of the art

$$\mathrm{CoSpan}\left(\mathrm{Stab}_{\infty}^{d\text{-CY}}(k((q)))\right)$$

Suitable ∞-category of cospans sent to Lagrangian correspondences

$$\mathrm{VHS}_{k((q))} \overset{\mathrm{HMS}}{\longleftrightarrow} \mathrm{GW}_{k((q))}$$

$$\mathrm{Lag}^{2-d}_{k((q))}$$

State of the art

[BD18]

$$\mathrm{CoSpan}\left(\mathrm{Stab}_{\infty}^{d\text{-CY}}(k((q)))\right)$$

$$\mathcal{M}_{-}$$

$$\mathrm{Lag}^{2-d}_{k((q))}$$

$$\mathrm{VHS}_{k((q))} \overset{\mathrm{HMS}}{\longleftrightarrow} \mathrm{GW}_{k((q))}$$

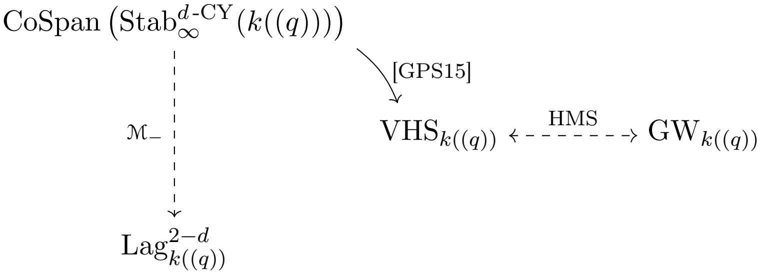
State of the art

$$\mathrm{CoSpan}\left(\mathrm{Stab}_\infty^{d\text{-CY}}(k((q)))\right) \xrightarrow{\mathcal{M}_-} \mathrm{Lag}^{2-d}_{k((q))}$$

$$\mathrm{VHS}_{k((q))} \xleftrightarrow{\text{HMS}} \mathrm{GW}_{k((q))}$$

[BBJ19]: Locally, a symplectic stack is the critical locus of a potential defined on a moduli of object

State of the art



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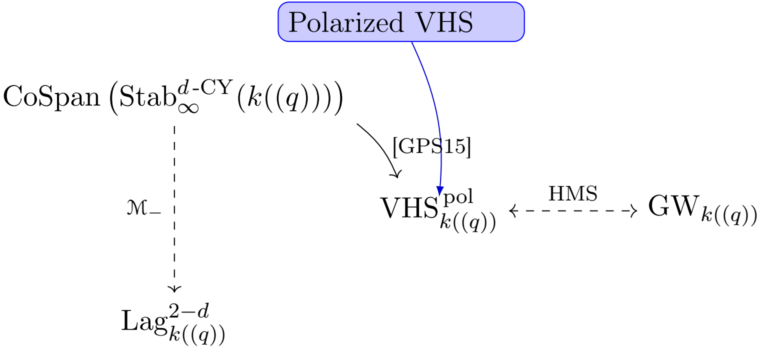

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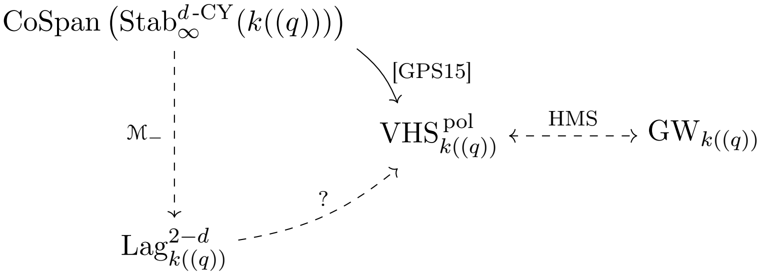

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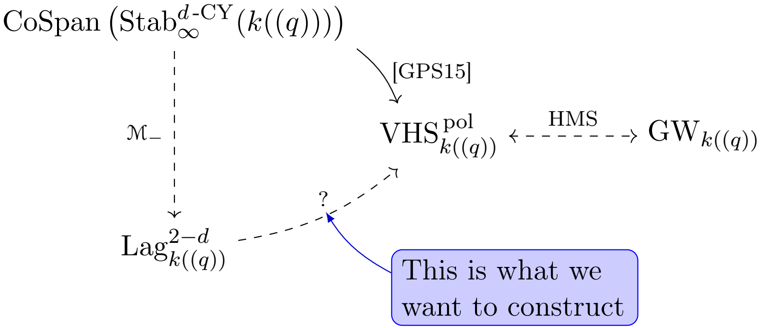

Thanks for your attention.

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