A very concise introduction to higher category theory and homotopical algebra

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INTRODUCTION: WHY DO WE NEED ∞-CATEGORIES?

Higher category theory is nowadays a fundamental language: its power and its versatility has been used extensively in the last thirty years of developments in mathematics. The language of higher categories, and especially the language of *∞-categories*, provides the apt formalism to study a wide array of homological and homotopical phenomena naturally arising in algebraic geometry, algebraic topology, commutative algebra, number theory, deformation theory, theoretical physics, symplectic geometry. Yet, its high complexity, together with the cult-like behavior of the mathematical community which naturally employs it[1], leads the "non-initiated" to look suspiciously at the idea of studying ∞-category theory, because they simply consider it to be a game which is not worth the candle. Indeed, the question that I get asked most frequently from algebraic geometers who do not use the higher categorical language is the following: *why* should I learn ∞-category theory? The goal of this introduction is to provide a (very partial) answer to this dilemma, showing some interesting examples which highlight how higher category theory naturally enters the game in algebraic topology and algebraic geometry, how it sheds a new epistemological light on many classical problems from both fields, and how it can be used to bypass many technical and theoretical drawbacks.

Why we need them in algebraic topology.

Ever since the beginning of time, man has yearned to study the homotopy category.

Montgomery Burns (more or less)

A rough, but truthful, description of the goals of algebraic topology could be the following: *algebraic topology is the study of algebraic invariants attached to topological spaces, which have to*

[1]In the preface to [Mum75], Mumford states that *«[algebraic geometry] seems to have acquired the reputation of being esoteric, exclusive, and very abstract, with adherents who are secretly plotting to take over all the rest of mathematics. In one respect this last point is accurate.»* This claim continues to be true if one replaces "algebraic geometry" with "higher category theory".

be sufficiently computable and powerful at the same time. The meaning of *sufficiently computable* is self-evident: we do not want a theory that makes it difficult to compute invariant even in the most elementary or interesting cases. The meaning of *sufficiently powerful* is a bit trickier: we want such invariants to be able to distinguish between spaces whenever they are "different" with respect to some fixed equivalence relation. The most natural relation one can impose is the one provided by the existence of homeomorphisms linking topological spaces, but this is *too* rigid. The second most natural one is the equivalence relation provided by the existence of *homotopy equivalences* linking topological spaces.

Definition (Homotopy equivalence)**.** Let X and Y be two topological spaces, let $f, g : X \to Y$ be two continuous maps between them.

(1) A *homotopy* between f and g is a map $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$. We denote the homotopy equivalence relation by $\simeq$.
(2) A map $f : X \to Y$ is a *homotopy equivalence* between X and Y if there exists a map $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

Most of the algebraic invariants that people encounter during their studies are indeed *homotopy invariant*, that is, they turn homotopy equivalences into strict isomorphisms of algebraic structures: see for example de Rham cohomology, singular homology and cohomology, cellular homology, fundamental groups and higher homotopy groups. (Some standard references for these theories are [BT82] and [Hat02].) This is a *so* fundamental feature of algebraic, and especially homological, invariants, that in 1945 Eilenberg and Steenrod explicitly *required* homology and cohomology theories to satisfy the homotopy invariance property ([ES45, Axiom 4]). Thus, it is natural to ask for the following: what if we worked directly with topological spaces considered only *up to homotopy equivalence*?

Definition. The *classical* (or *naive*) *homotopy category* Ho(Top) is the category obtained from the category Top of topological spaces in the following way.

(1) The class of objects of Ho(Top) is the same as the class of objects of Top.
(2) The set of maps between two topological spaces X and Y is the quotient set

$$[X, Y] \coloneqq \mathrm{Hom}_{\mathrm{Top}}(X, Y)/\simeq .$$

It turns out that, in general, this is actually the *wrong* definition of a homotopy category of topological spaces. Indeed, we are mainly interested in studying the *homotopy type* of a space X – that is, its homotopy groups $\pi_n(X, x)$ for any choice of a base point x. Yet, in Ho(Top) we can find topological spaces sharing the same homotopy type which however are not isomorphic.

Example. Let L be the *long line*, defined as follows. Letω_1 be the first uncountable ordinal, and consider the totally ordered set

$$R \coloneqq \omega_1 \times (0,1]$$

equipped with the lexographic order: $(\alpha, t) < (\beta, s)$ if and only if $\alpha < \beta$ or $\alpha = \beta$ and $t < s$. This order induces a topology (the *order topology*), admitting a basis of open subsets of the form

$$((\alpha, t), (\beta, s)) \coloneqq \{(\gamma, u) \in R \mid (\alpha, t) < (\gamma, u) < (\beta, s)\} .$$

Taking R^{op} to be the same set as R equipped with the reverse order, then L is defined as

$$L := \left(R \coprod R^{\text{op}}\right) / \sim$$

where $\sim$ glues $(0,1) \in R^{\text{op}}$ (its maximum) with $(0,0) \in R$ (its minimum). L is connected, and for any choice of a point $l \in L$ and for any $n \in \mathbb{N}$ one has

$$\pi_n(L,l) \cong \{*\}.$$

Yet, the inclusion $\iota\colon \{l\} \hookrightarrow L$ is *not* a homotopy equivalence, in spite of inducing isomorphisms on every homotopy group: in particular, in Ho(Top) we lack the inverse of (the homotopy equivalence class of) the inclusion ι. (Let us remark that this issue does not arise if X and Y are both CW-complexes and there exists a map $f : X \to Y$ inducing isomorphisms on all homotopy groups, in virtue of the Whitehead theorem: see for example [Hat02, Chapter 4, Theorem 4.5]).

What we actually need to do in order to really identify topological spaces with the same homotopy type is to *localize* the category Top at the class $\mathscr{W}$ of *weak* homotopy equivalences. This procedure formally inverts the maps that induce isomorphisms on all homotopy groups via the so called *calculus of fractions* (see [GZ67]). This machinery has fundamental size issues for a general category $\mathscr{C}$ with a given choice $\mathscr{W}$ of a class of weak equivalences; yet, these issues do not arise when $\mathscr{C}$ admits a *model category structure* (see the original, enlightening, book [Qui67], or [Hov99] for a standard and modern reference). This is the case for Top, and so we have a well defined homotopy category hTop. Still, things are quite messy.

Example. Consider the multiplication map $S^1 \xrightarrow{\cdot 2} S^1$ defined as $e^{i\theta} \mapsto e^{2i\theta}$. We are interested in understanding what should be the correct colimit, in the homotopy setting, of the following diagram.

$$\begin{tikzcd} S^1 \arrow[r, "\cdot 2"] \arrow[d, "p"'] & S^1 \\ \{*\} & \end{tikzcd}$$

(1) In Top, the pushout is modeled by the coproduct $S^1 \coprod \{*\}$ modulo the equivalence relation generated by identifications $e^{i\theta} \simeq \{*\}$ whenever there exists a $\varphi \in [0, 2\pi]$ such that $e^{i\theta} = e^{2i\varphi}$ and $\{*\} = p\left(e^{i\varphi}\right)$. In particular, since both $\cdot 2$ and p are surjective, it follows that $\{*\}$ is in relation with all the points in S^1, hence the *strict* pushout is just a singleton. We have now a induced diagram of fundamental groups

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z} \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}$$

which is however *not* a pushout of groups. In some sense, the strict pushout in Top does not behave well with respect to the underlying homotopy type.

(2) In hTop, the induced diagram *does not even admit a pushout*! This is a standard computation in algebraic topology: first, observe that if Q was a pushout in hTop, then for any

other pointed space (Z, z), the maps (up to homotopy) from Q to Z would compute the elements of order 2 inside $\pi_1(Z, z)$. So by taking a fibration sequence of pointed topological spaces

$$X \longrightarrow Y \longrightarrow Z$$

one would get a long exact sequence of generalized homotopy groups (see [May99, Chapter 6])

$$\ldots \to \left[Q, \Omega^2 Z\right] \to [Q, \Omega X] \to [Q, \Omega Y] \to [Q, \Omega Z] \to [Q, X] \to [Q, Y] \to [Q, Z].$$

Here, $\Omega^n X$ is the n-fold space of loops based at the base point x of X. But from the Hopf fibration

$$S^1 \longrightarrow S^3 \longrightarrow S^2$$

we get, after modding out the action of S^0, a fibration

$$S^1 \longrightarrow \mathbb{P}^3_{\mathbb{R}} \longrightarrow S^2.$$

Taking the maps up to homotopy equivalence from Q, using that $\pi_1(S^1) \cong \mathbb{Z}$, $\pi_1(S^2) \cong 0$, and $\pi_1(\mathbb{P}^3_{\mathbb{R}}) \cong \mathbb{Z}/2\mathbb{Z}$, we get the sequence

$$\ldots \longrightarrow 0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \longrightarrow \ldots$$

which is obviously non exact. This rules out the existence of such Q in hTop.

(3) Finally, we could try a more sophisticated approach. Recall that in concrete categories, in order to compute the classical pushouts over a span

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Z \\ Y & \end{tikzcd}$$

we first build a coproduct of Y and Z, and then mod out by the relation that identifies elements $y \in Y$ and $z \in Z$ whenever they are the image of the same element $x \in X$ via f and g, respectively. We can picture this idea by saying that $y \simeq z$ in $Y \coprod_X Z$ if and only if there is a diagram of identifications

$$\begin{tikzcd} & x \arrow[dl, "f"'] \arrow[dr, "g"] & \\ y & & z. \end{tikzcd}$$

The collection of all these diagrams is a, possibly very convoluted, graph; yet, classical pushouts only recover its connected components. But what if we *do not forget* about these paths that connect elements from X, Y, and Z? We get a way finer topological object, which has not only a set of connected components but also paths, 2-cells, and so forth, which define a *structure* that remembers how the *property* of "being equal" holds between elements. (To read more about this presentation of homotopy colimits, see the excellent slides by Anel [Ane18].)

For topological spaces, in particular, a method to implement the above idea would be to consider the disjoint union of the two topological spaces Y and Z (in our case

case, a circle S^1 and a singleton $\{*\}$) and of the cylinder $X \times \mathbb{I}$ over X (so, a honest cylinder $S^1 \times \mathbb{I}$). If we image X to be embedded in $X \times \mathbb{I}$ via the inclusion $x \mapsto \left(x, \frac{1}{2}\right)$, we can interpret our cylinder as a space parametrizing paths out of X in both directions. Finally, we glue these three topological in the following way: we identify an element $(x, 0) \in X \times \mathbb{I}$ with $f(x) \in Y$, and points $(x, 1) \in X \times \mathbb{I}$ with $g(x) \in Z$. In our case, this is just the standard presentation of the *mapping cone* of the double covering map $S^1 \xrightarrow{\cdot 2} S^1$, see for example [Hat02, Page 13]. This is the notion that makes the most sense in the homotopical setting: for example, one can prove that this object indeed has trivial homotopy groups except for the fundamental group, which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, which is precisely what we needed in order to make the long sequence of homotopy groups *exact*. However, this construction still has some drawbacks: it has no immediate universal property, and does not agree with either the colimit in Top or in hTop. Rather, it should be a "higher colimit" in a more natural environment.

A place where to study cohomological theories and stable homotopy theory. Homology and cohomology theories, in order to be meaningful, are supposed to satisfy the *Eilenberg-Steenrod axioms*. Among requirements concerning homotopy invariance, the existence of Mayer-Vietoris sequences, and excision formulae, Eilenberg and Steenrod included also the following *axiom of dimension*: any homology theory $E_\bullet$ in degree $n \neq 0$ should be trivial when evaluated on the point, i.e., $E_n(\{*\}) \cong 0$ for all $n \neq 0$; the same has to hold for cohomology theories. By erasing this axiom, one gets *generalized* homology and cohomology theories, such as *algebraic K-theory* ([Qui73]) and *complex cobordism* ([Ati61]). It turns out that these generalized cohomology theories inhabit a no man's land sitting between algebra and topology, in the following sense.

(1) By the Brown Representability Theorem ([Bro62, Theorem 1]), any cohomology theory functor

$$E^n \colon \mathrm{hTop}^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

is representable by (the representative of) a CW complex, denoted by $K(E, n)$. Moreover, Eilenberg-Steenrod axioms imply that for any X there exists an isomorphism $E^n(X) \cong E^{n+1}(\Sigma X)$, where Σ is the suspension functor of pointed topological spaces. In virtue of the adjunction $\Sigma \dashv \Omega$, one has a chain of isomorphisms

$$[X, K(E,n)] \cong E^n(X) \cong E^{n+1}(\Sigma X) \cong [\Sigma X, K(E, n+1)] \cong [X, \Omega K(E, n+1)],$$

which yields an isomorphism in hTop (i.e., a *real* homotopy equivalence in Top) between $K(E, n)$ and $\Omega K(E, n+1)$. By the very same adjunction, the homotopy equivalences $K(E, n) \xrightarrow{\simeq} \Omega K(E, n+1)$ yield structure maps $\Sigma K(E, n) \to K(E, n+1)$ for any non-negative integer n.

(2) On the converse, for any topological space we can consider its n-fold suspension $\Sigma^n X$, and define a collection of topological spaces

$$\Sigma^\infty X := \{\Sigma^n X\}_{n \in \mathbb{N}},$$

with structure maps $\Sigma(\Sigma^\infty X_n) \to \Sigma^\infty X_{n+1}$ given by the natural homeomorphism $\Sigma(\Sigma^n X) \cong \Sigma^{n+1} X$. Taking the set of functors

$$[-, \Sigma^n X] : \mathrm{hTop} \longrightarrow \mathrm{Set}$$

produces a generalized cohomology theory.

Since the reduced suspension functor Σ is equivalent to taking the smash product of pointed topological spaces with S^1, where the smash product of two pointed topological spaces (X, x) and (Y, y) is defined as

$$(X, x) \wedge (Y, y) := ((X \times Y) / (X \vee Y), (x_0, y_0)),$$

we can abstract these key ideas as follows.

Definition ([Lim58; Ada74]). A *topological spectrum* $E^\bullet$ is a collection of pointed topological spaces $\{E^n\}_{n\in\mathbb{N}}$ together with maps $S^1 \wedge E^n \to E^{n+1}$.

Actually, we have just defined topological *sequential* spectra, but there are many other sligthly different notions that shape the same underlying, homotopical theory: topological Ω-spectra, sequential spectra in simplicial sets, excisive functors from finite pointed simplicial sets to pointed simplicial sets, combinatorial spectra, and so on. All of these objects are gathered in appropriate categories which moreover admit model structures, and all of these categories are linked by (a zig zag of) Quillen equivalences, hence they present the same homotopy category, which is usually called *stable homotopy category*, or just *stable category*, and is denoted by hSp. The stable homotopy category is equipped with a ∞-*loop space* functor

$$\Omega^\infty : \mathrm{hSp} \longrightarrow \mathrm{hTop},$$

which agrees with taking the standard free ∞-loop space construction for a pointed topological space

$$QX := \operatorname*{colim}_{k\to\infty} \Omega^k \Sigma^k X.$$

Yet, in hSp, the ∞-loop space is the right adjoint in an adjoint equivalence, where the left adjoint is the *suspension spectrum functor*

$$\Sigma^\infty : \mathrm{hTop} \longrightarrow \mathrm{hSp}.$$

The fact that taking loops becomes an invertible operation in hSp makes this category the natural environment where to study *stable homotopy groups of topological spaces*, i.e., the homotopy groups

$$\pi_n^s(X) := \operatorname*{colim}_{k\in\mathbb{N}} \pi_{n+k}(\Sigma^k X).$$

These stable homotopy groups have been widely studied because of the discovery of some important results concerning the stabilization of homotopy groups of n-connected topological spaces, such as the *Freudenthal Suspension Theorem* ([Fre38; Swi75]).

The stable homotopy category hSp enjoys many other nice properties. It admits a symmetric monoidal structure given by the *smash product of spectra*, with monoidal unit given by the *sphere spectrum* $\mathbb{S} := \Sigma^\infty S^0$, and there is a natural transformation

$$\Sigma^\infty (X \wedge Y) \longrightarrow \Sigma^\infty X \wedge \Sigma^\infty Y$$

which agrees with the symmetric monoidal structures on both hTop and hSp. In particular, one can think of the stable homotopy category as a natural environment where to study "non-linear homological algebra": spectra play the role of abelian groups in the topological/homotopical setting, and taking monoids with respect to the smash product produces homotopy-coherent algebraic structures called *ring spectra*, which recover ordinary homological algebra by considering the cohomology theory with coefficients in any commutative ring. Yet, *no model category* can present the stable homotopy category and enjoy *strict* variants of *all* the above properties (i.e., monoidality under smash product, adjunction between Σ^∞ and Ω^∞, etc.): for reference, see the landmark paper [Lew91]. Given how badly homotopy categories of model categories behave – they are not even *concrete* categories, see the original paper for topological spaces [Fre70] and its recent generalization [LL18] – this is a huge problem if one wants to study of algebraic topology by the means of ring spectra.

Why we need them in algebraic geometry. Historically, homological algebra arose thanks to the interest and efforts by Riemann, Betti, Poincaré, Noether (who first realized that Betti numbers actually computed the rank and the torsion of some graded abelian group), de Rham, and a lot of other mathematicians who wanted to study manifolds and topological spaces between the ending of the 19th century and the first half of the 20th century: a beautiful account of the history of its developments is to be found in [Wei99]. Yet, it was only between 1950 and 1956 that Cartan and Eilenberg wrote a book which extracted the main algebraic and theoretical (now we would say: *homological*) tools out of the machinery usually applied to topological spaces, coining the phrase *homological algebra* (which was moreover chosen as the title for their book [CE56]). Immediately, Grothendieck and all the mathematicians belonging to European algebraic geometry school appropriated this formalism in order to study problems arising from complex algebraic geometry, number theory and commutative algebra. One of the most important results in this direction, nowadays hailed as the beginning of modern derived algebraic geometry, is the following.

Theorem (Serre's intersection formula, [Ser65])**.** *Let Y and Z be two closed subschemes of an ambient space X, cut out by two sheaves of ideals $\mathscr{I}_Y$ and $\mathscr{I}_Z$, respectively. Then the intersection multiplicity at a point x in the set-theoretic intersection $Y \cap X$ is computed by the Euler characteristic*

$$m := m(x, Y, Z) = \sum_{i \geqslant 0} (-1)^i \operatorname{length}_{\mathscr{O}_{X,x}} \left(\operatorname{Tor}_i^{\mathscr{O}_X} (\mathscr{O}_X / \mathscr{I}_Y, \mathscr{O}_X / \mathscr{I}_Z) \right).$$

The groundbreaking idea of this formula is that, if Y and Z do not meet transversally and they are not at least Cohen-Macaulay, the merely scheme-theoretic data are not enough to compute the multiplicity, for which one needs to taking into account some higher order corrections of homological nature. Similar ideas and challenges, such as the need for an extension of Serre duality to arbitrary proper schemes over a base, led soon to the development of *derived categories of quasi-coherent sheaves* on schemes, and in general *derived categories of abelian categories*, where homological operations such as deriving left and right exact functors naturally take place. The derived category $h\mathscr{D}(\mathscr{A})$ of an abelian category $\mathscr{A}$ was introduced by Verdier in his PhD thesis [Ver96], and even if it was not yet apparent, such construction was actually the shadow of taking the homotopy theory of a model structure bestowed on category of chain complexes of objects belonging to $\mathscr{A}$. Differently from the homotopy category of topological spaces, however, $h\mathscr{D}(\mathscr{A})$ is naturally a *triangulated category*, i.e., it admits a collection of *exact triangles* which generalize and axiomatize the ideas of long exact sequences in cohomology and mapping cones. This theory has famously a huge drawback: the cone which extends any morphism in $h\mathscr{D}(\mathscr{A})$ to an exact triangle is unique up to a *non-unique* isomorphisms, which leads to some unpleasant consequences.

Example. Fix a field $\Bbbk$, and consider in $h\mathscr{D}(\Bbbk)$ the following map of arrows.

$$\begin{tikzcd}
\Bbbk \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \Bbbk[1]
\end{tikzcd}$$

Then a mapping cone for the first arrow is $\Bbbk[1]$, and a mapping cone for the second arrow is $\Bbbk[1]$ as well. Yet, there is not a unique choice of a map $\Bbbk[1] \to \Bbbk[1]$ which produces a map of exact triangles.

$$\begin{tikzcd}
\Bbbk \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \Bbbk[1] \arrow[r, "\cong"] \arrow[d, shift right, "0"'] \arrow[d, shift left, "\cong"] & \Bbbk[1] \arrow[d] \\
0 \arrow[r] & \Bbbk[1] \arrow[r, "\cong"] & \Bbbk[1] \arrow[r] & 0.
\end{tikzcd}$$

Actually, there is more: there is an *insurmountable* obstruction to choosing a canonical mapping cone of a morphism in the derived category.

Theorem ([Ver96, Proposition II.1.2.13])**.** *Let $\mathscr{T}$ be a triangulated category where countable products and countable direct sums exist. Let* $\mathrm{Arr}(\mathscr{T})$ *be the category of morphisms in $\mathscr{T}$, and let* $\mathrm{Triang}(\mathscr{T})$ *be the category of exact triangles n $\mathscr{T}$, and suppose there exists a functor*

$$\mathrm{cone}\colon \mathrm{Arr}(\mathscr{T}) \longrightarrow \mathrm{Triang}(\mathscr{T}).$$

Then every trangle in $\mathscr{T}$ splits, i.e., is isomorphic to a triangle of the form

$$X \to X \oplus Y \to Y \xrightarrow{0} X[1].$$

This issue, in the case of $h\mathscr{D}(\mathscr{A})$, can be explained with the following slogan: $h\mathscr{D}(\mathscr{A})$ is the homotopy category of the category $C_\bullet(\mathscr{A})$ of chain complexes in $\mathscr{A}$, but $\mathrm{Arr}(h\mathscr{D}(\mathscr{A}))$ is *not* the homotopy category of the category $\mathrm{Arr}(C_\bullet(\mathscr{A}))$, with some induced model structure. Another problem, which is even more severe, is the following: derived categories do not satisfy descent.

Example. Fix again a field $\Bbbk$ and consider the usual presentation of $\mathbb{P}^1_\Bbbk$ as the gluing of two copies of $\mathbb{A}^1_\Bbbk$ along $\mathbb{G}_{m,\Bbbk}$. In particular, we have a presentation of $\mathbb{P}^1_\Bbbk$ as a pushout in schemes:

$$\mathbb{P}^1_\Bbbk \cong \mathbb{A}^1_\Bbbk \coprod_{\mathbb{G}_{m,\Bbbk}} \mathbb{A}^1_\Bbbk.$$

Applying the contravariant functor $\mathscr{D}$, we have an induced square of derived categories.

$$\begin{tikzcd} \mathrm{h}\mathscr{D}\left(\mathbb{P}^1_{\Bbbk}\right) \arrow[r] \arrow[d] & \mathrm{h}\mathscr{D}\left(\mathbb{A}^1_{\Bbbk}\right) \arrow[d] \\ \mathrm{h}\mathscr{D}\left(\mathbb{A}^1_{\Bbbk}\right) \arrow[r] & \mathrm{h}\mathscr{D}\left(\mathbb{G}_{\mathrm{m},\Bbbk}\right) \end{tikzcd}$$

This diagram is *not* a pullback square. More precisely: the natural functor

$$\mathrm{h}\mathscr{D}\left(\mathbb{P}^1_{\Bbbk}\right) \longrightarrow \mathrm{h}\mathscr{D}\left(\mathbb{A}^1_{\Bbbk}\right) \times_{\mathrm{h}\mathscr{D}\left(\mathbb{G}_{\mathrm{m},\Bbbk}\right)} \mathrm{h}\mathscr{D}\left(\mathbb{A}^1_{\Bbbk}\right)$$

is essentially surjective but not fully faithful.

The *leitmotiv* of these examples could be summarized as follows: it is not enough to know that two (complexes of) quasi-coherent sheaves defined on two open subsets agree on their intersection up to homotopy, but we also need to remember *the homotopy itself*. An analogous claim holds for morphisms between them. These drawbacks can be solved by replacing the ordinary derived category $\mathrm{h}\mathscr{D}(\mathscr{A})$ with some ∞-categorical enhancement $\mathscr{D}(\mathscr{A})$ whose homotopy category is precisely $\mathrm{h}\mathscr{D}(\mathscr{A})$. This, *a posteriori*, justifies our choice of notation for the ordinary derived category of $\mathscr{A}$.

Another important topic of research in algebraic geometry which eventually led to the development of higher categorical formalism is the study of *moduli problems* arising from algebraic and arithmetic contexts. A moduli problem for a category $\mathscr{C}$, in the most naive sense, can be simply described as a contravariant functor $\mathscr{M}\colon \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$. Given an object C in $\mathscr{C}$, the set $\mathscr{M}(C)$ has to be though as a set of objects or structures depending naturally (in the categorical sense) on C: for simplicty, from now on we shall call an element in $\mathscr{M}(C)$ as a *family over* C. For example, when $\mathscr{C}$ is the category Sch of schemes, $\mathscr{M}$ can parametrize the isomorphism classes of vector bundles, or the isomorphism classes of principal bundles with respect to a smooth group scheme G, or the isomorphism classes of smooth curves of fixed genus g and n marked points (for the case $g = n = 1$, this produces the moduli problem parametrizing elliptic curves). If moreover the moduli problem $\mathscr{M}$ is representable, i.e., there exists an object $X_{\mathscr{M}}$ such that for any other C in $\mathscr{C}$ there is a natural bijection

$$\mathscr{M}(C) \cong \mathrm{Hom}_{\mathscr{C}}(C, X_{\mathscr{M}})$$

then the complexity of the moduli problem lowers considerably. Again, for $\mathscr{C} = \mathrm{Sch}$, this means that there is some honest scheme which classifies *completely* the geometric structures we are interested in, and (in virtue of the fully faithfulness of the Yoneda embedding) studying $X_{\mathscr{M}}$ is *the same* as studying its associated moduli problem $\mathscr{M}$. Moreover, an immediate consequence of the representability of $\mathscr{M}$ is that any family $\mathscr{F} \in \mathscr{M}(S)$ over an arbitrary scheme S comes from a *universal family* over $X_{\mathscr{M}}$, i.e., there exists a universal element

$$\mathscr{U} \in \mathscr{M}(X_{\mathscr{M}}) \cong \mathrm{Hom}_{\mathrm{Sch}}(X_{\mathscr{M}}, X_{\mathscr{M}})$$

corresponding to the identity automorphism of $X_{\mathscr{M}}$, such that for any other scheme S and any family $\mathscr{F}$ in $\mathscr{M}(S)$, there exists a unique morphism $\chi : S \to X_{\mathscr{M}}$ such that $\mathscr{F}$ is the image of $\mathscr{U}$ under $\mathscr{M}(\chi)\colon \mathscr{M}(X_{\mathscr{M}}) \to \mathscr{M}(S)$. One of the most classical examples of this phenomenon is the absolute n-dimensional projective space $\mathbb{P}^n_{\mathbb{Z}}$, which represents the moduli problem

$$X \mapsto \{\text{line bundles } \mathcal{L} \text{ over } X \text{ with a choice of a basis } \langle s_0, \dots, s_n \rangle \text{ of its global sections}\}.$$

In this case, the universal line bundle corresponds to the tautological line bundle $\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(-1)$.

However, many interesting moduli problems fail to be representable. In practice, many *families* over an object C are honest objects $\mathcal{F} \to C$ living over C, and the condition of the universality of the family $\mathcal{U} \to X_{\mathcal{M}}$ boils down to the condition that for any family $\mathcal{F}$ there exists a unique $\chi_{\mathcal{F}}$ such that

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d, "f"'] & \mathcal{U} \arrow[d] \\ C \arrow[r, "\chi_{\mathcal{F}}"'] & X_{\mathcal{M}} \end{tikzcd}$$

is a pull-back diagram and $\chi_{\mathcal{F}} \circ f$ corresponds to the image of $\mathcal{U}$ under the map $\mathcal{M}(\chi_{\mathcal{F}}) \colon \mathcal{M}(X_{\mathcal{M}}) \to \mathcal{M}(C)$. So, whenever there is some S such that $\mathcal{M}(S)$ admits a non-trivial automorphism

$$\alpha \colon \mathcal{M}(S) \xrightarrow{\cong} \mathcal{M}(S)$$

fixing *any* family $\mathcal{F} \in \mathcal{M}(S)$, there is an obstruction to the representability of the moduli problem: this automorphism fixing $\mathcal{F}$ produces a *non-trivial isotrivial family* (that is, another family with isomorphic fibers). This problem arises, for example, in the case of moduli of elliptic curves ([KM85]) and more generally in a wide class of moduli of curves ([HM98]). In order to solve this issue, one can work in many different ways.

(1) One can settle for only *coarse* moduli space: this is the universal object X_M which is equipped with a natural transformation of functors $\mathcal{M} \to \mathrm{Hom}_{\mathcal{C}}(-, X_M)$, which is however not required to be an isomorphism anymore. For example, the moduli problem of elliptic curves over a ring R admits a *coarse* moduli space given by the affine line $\mathbb{A}^1_R$: indeed, any elliptic curve is determined by its j-invariant, which is a scalar in R. This is, however, not fine enough to completely capture transformations of elliptic curves.
(2) Alternatively, one could add constraints in order to force such automorphisms to disappear. In the case of elliptic curves one could for example ask also for some *level n structure* (see [Dri74] or the account in [KM85, Chapter 5]).
(3) Finally, one could give up asking for a moduli problem *with values in sets* and keep track of every possible automorphism of $\mathcal{M}(C)$, yielding a *groupoid* instead. The moduli problem can then be studied with the theory of *algebraic stacks* ([Gir65; DM69]).

The classical definition of a stack is given in terms of a fibration in groupoids $\mathcal{S} \to \mathrm{Sch}$ over the category of schemes: such fibrations are then supposed to satisfy some descent data that allows to glue the groupoid corresponding to the fiber $\mathcal{S}_X$ over any scheme X along the fibers $\mathcal{S}_{U_i}$ over any étale covering $\{U_i \to X\}_{i \in I}$. This definition, in virtue of the Grothendieck construction ([Tho79]), leads to the following principle: stacks should be nothing more than *étale* 2-*sheaves* *in* 1*-groupoids*, i.e., higher sheaves for the étale topology on schemes with values in the 2-category of groupoids. This idea becomes a proper and well-tuned definition in the context of ∞-categories.

Remark. The above list is far, *far* from being complete, of course. For instance, another historically and philosophically relevant thread of research worth mentioning is represented by *deformation theory*, which arose in the mid Fifties thanks to the work of Kodaira and Spencer ([KS58]), who proved a relation between first order deformations of smooth projective complex manifolds and first homology classes in classes with coefficients in the holomorphic tangent sheaf T_X in $\mathrm{H}^1(X, \mathrm{T}_X)$. Later, following the philosophy that the tangent sheaf should control deformation and obstruction theory, Quillen developed a homology theory for commutative rings (*André-Quillen homology*) in [Qui70], which was later globalized in [Ill71; Ill72], replacing the module of Kähler differentials with a more well-behaved object: the *cotangent complex*. The close relationship between derived algebraic geometry and deformation theory was then hinted in a now famous letter by Drinfeld to Schechtman ([Dri14]), which coined the *derived deformation principle* which laid the foundations for almost all subsequent research in the topic of *derived deformation theory*: the reader can find in [DSV22] an incredibly well written account of the Maurer-Cartan methods that led to the derived deformation principle. Let us finally remark that this principle is *false* and very ill-behaved if one takes into account *non-derived* deformation theory. The interested reader can delve into this theory, and many others as well, in fuller detail by reading Toën's recent and beautiful survey on derived algebraic geometry [Toë14].

How algebraic geometry and algebraic topology interact. In virtue of the analogies between the homotopy category of topological spaces and the derived category of an abelian category of modules, it has been long conjectured that homotopy theory should provide a common ground where to study both algebra and topology. In particular:

(1) One should be able to consider for any non negative integer n a theory of n*-stacks* – which, roughly speaking, should be *sheaves for some suitable Grothendieck topology with values in homotopy types of topological spaces whose higher homotopy groups are trivial in degrees* $m > n$. This is linked to the theory of ordinary stacks in virtue of the *homotopy hypothesis*: weak n-groupoids (that is, groupoids with non-trivial k-arrows for every degree up to n, with composition of morphisms defined up to some given homotopy-coherent datum) should be modeled by n-homotopy types, see for example [GJ99, Theorem 11.4] or [May67, Chapter III, §16]. In particular, ordinary algebraic stacks are stacks of 1-homotopy types, hence 1-stacks.
(2) One should be able to consider *affinization of homotopy types*, that is, for any reasonable topological space X there should be some related algebraic scheme or stack from which one could recover the homotopy type of X, and this association should be natural in X.

These ideas were envisioned (among others) by Grothendieck himself, who sketched them out in a decent amount of details in a very influential letter to Quillen in 1983 ([Gro22]). To highlight how far-reaching his insight was, let us recall that Voevodsky himself acknowledged Grothendieck's influence when he started developing *motivic homotopy theory*, which was a natural reinterpretation of the homotopy of topological spaces and higher categories from the perspective of algebraic geometry and commutative algebra. Moreover, the problem of affinization of homotopy types is still being investigated, see for example one of the latest contribution to this subject by Toën in [Toë20].

What we want to do. Nowadays, derived algebraic geometry has moved towards directions and concepts that go far beyond the scopes briefly presented above – and this happened also *because of* the power and the versatility of the formalism of higher category theory. For example, the most recent formulations of the mirror symmetry hypothesis in physics (from Kontsevich [Kon95] onwards) are given in terms of some equivalence between A_∞*-categories* (or *differential graded categories*), which are in turn a higher enhancement of the concept of triangulated categories. So, to sum up the content of this introduction: an algebraic geometer does not have to completely grasp the theory of higher categories and derived geometry; yet, there are plenty of reasons why an algebraic geometer should be interested in understanding them and look at them with less suspicion. The problem is that higher category theory is indeed a *hard* subject: differently from ordinary category theory, constructions have to be carried out using universal properties and formal existence results, rather than by the means of explicit computations and definitions "on the nose". This happens because ∞-categories rely on an *infinite amount of homotopy coherence data*: we are relaxing the concept of "equality" asking objects to be only "equivalent", thus for everything to work properly one needs to specify not only equivalences between objects, but also homotopies between maps, 2-homotopies between homotopies, and so forth.

The goal of these short lectures is the following: we want to provide the reader an intuition on how to deal with ∞-categories; to teach them how to distinguish those cases in which a "1-categorical arguments" can be adapted almost *verbatim* in the ∞-categorical world, and those in which one has to pay some caution to more subtle details; to explain how to do explicit computations with strict models; and finally, hopefully, to allow them to translate freely notions and concepts borrowed from ordinary triangulated categories, dg categories, and stable ∞-categories. Most of all, we want to convince them that the apparently insurmountable challenge of specifying an infinite amount of homotopy coherence is a very little price to pay for a beautiful, rich and smooth theory: with some training, it is even *natural* to think about certain topics directly in the ∞-categorical world.

To sum it up: these notes want to present briefly some key ideas in the development of homotopical algebra and higher category theory, and aim to provide some sort of *cheat sheet*, allowing the reader to avoid the most technically demanding and challenging parts of books such as *Higher Topos Theory* [Lur09] or *Higher Algebra* [Lur17], extracting the most fundamental algebraic ideas behind the simplicial and quasi-categorical machinery, and providing some less esoteric perspectives on how to deal with ∞-categories, stable ∞-categories, ring spectra which are linear over $\mathbb{Z}$ and commutative ring spectra which are linear over a base ring of characteristic 0. Because of this, these notes critically lack proofs but have *a lot* of references where to check more general statements and proofs.

Prerequisites for these notes. These notes are thought for an audience that, albeit not knowledgeable of homotopy theory, derived geometry and higher category theory, still has some background in algebraic geometry and homological algebra *à la Grothendieck*. In particular, I assume the reader to know some commutative algebra (up to Kähler differentials), some very basic theory of derived categories *à la Verdier*, the theory of derived functors between abelian categories in terms of δ-functors computed on projective or injective resolutions. Of course, I assume some basic knowledge of the theory and the concepts of ordinary category theory one can encounter in a standard Master course (definitions, limits, colimits, adjunctions, equivalences of categories, Yoneda lemma, monoidal categories, abelian categories...).

Some basic references. Nowadays ∞-category is a language freely employed by a large number of mathematicians coming from the most disparate fields of research. As such, plenty of references have started to spring out here and there, more or less user-friendly. Among the most useful ones that I both know *and use* I can pinpoint at least:

(1) Lurie's books *Higher Topos Theory* [Lur09] and *Higher Algebra* [Lur17], which still are some of the most important books (if not *the* most important books) on the subject. Unfortunately, they are *not* user-friendly at all: they have been written bearing in mind the necessities of algebraic topologists, and the formalism and the strategy of the proofs can reach really convoluted heights. Nowadays, to learn the basics of higher category theory there are many other references which look more accessible and compelling than *Higher Topos Theory* for a novice reader (see below). On the other hand, *Higher Algebra* is still the most complete account on the subject of higher categorical algebra one can find, with precise and incredibly general statements of *lots* of both technical and conceptual results: it still the most indispensable source on the subject. (Actually, the amount and the precision of the results in both books are *precisely* the reasons why they are still mistaken for introductory writings to the topic of higher categorical algebra in the first place.)

(2) A really nice reference for the foundational theory of ∞-category, with carefully spelled statements and proofs (and a rather pleasant looking style) is to be found in [RV20] and especially in [RV22]. When compared to [Lur09], these ones are *way* more accessible and clearly intended to be read by an audience accustomed only to classical category theory. The latter is particularly recommended.

(3) A more concise and gentler introduction to higher categorical algebra and homotopical algebra, heavily relying on Lurie's work, is to be found in [Gep20].

(4) Finally, recently Land published a book [Lan21], which aims to be some self-contained, undergraduate-friendly reference for the foundational theory of ∞-categories. It also features some interesting exercises at the end, if one is interested in rolling up their sleeves and do some computations by themselves.

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1. INTRODUCTION TO HOMOTOPICAL CATEGORY THEORY: MODEL CATEGORIES

There are many different reasons why an introductory course on homotopical algebra and higher category theory should start with *model categories* – other than their historical significance for the subject, of course.

(1) The most fundamental example of "homotopy categories" – the homotopy category of topological spaces and the derived category of an abelian category – come from model categories, and having an even superficial knowledge of their axioms and properties is quite useful for computations.

(2) One of the most fundamental results about model categories, namely the Quillen equivalence between topological spaces and simplicial sets, hinted at one of the main concepts of homotopy theory: namely, that *simplicial* leads to *homotopy-theoretic*. This is the reason why Boardman and Vogt introduced *weak Kan complexes* in 1973 ([BV73]). These objects, under the name *quasi-categories*, were later studied by Joyal starting from the 1980s ([Joy02]), and are precisely the model for ∞-categories used by Lurie in [Lur09].

(3) Finally, we shall see that many important ∞-categories that we will be interested in arise naturally as a *Dwyer-Kan localization* of *simplicial combinatorial model categories*, in a unique way up to a zig zag of Quillen equivalences. At this point, this is probably nothing more than gibberish for the reader, but the power of this statement will hopefully be made clearer later.

References for this section. The theory of model categories is now very well understood: the original article [Qui67] in which Quillen first introduced this concept is still very accessible to the modern reader (up to some slight modifications of the original definition, see for instance [Qui69]), but Hovey's book [Hov99] arguably wins the cake as the most "classical" reference for this subject. Notice that there is now a LaTeXversion of **??**: this makes it a *mandatory* reading – just as almost everything Quillen wrote.

1.1. **Definitions, main properties and constructions.**

In order to give the correct notion of a model structure on a category, we need the following preliminary definitions.

Definition 1.1.1. Let $\mathscr{C}$ be an ordinary category, and let $\mathrm{Arr}(\mathscr{C})$ be the category of its arrows.

(1) A map $f: X \to Y$ is a *retract* of another map $g: U \to V$ if f is a retract of g in the category $\mathrm{Arr}(\mathscr{C})$, i.e., if there exists a commutative diagram as follows.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}_X"] & U \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r] \arrow[rr, bend right, "\mathrm{id}_Y"'] & V \arrow[r] & Y \end{tikzcd}$$

(2) A *functorial factorization* is an ordered pair of functors

$$(\alpha, \beta) \colon \mathrm{Arr}(\mathscr{C}) \times \mathrm{Arr}(\mathscr{C}) \longrightarrow \mathrm{Arr}(\mathscr{C})$$

such that any morphism $f \colon X \to Y$ in $\mathscr{C}$ admits a factorization $f = \beta(f) \circ \alpha(f)$.

(3) Given two maps $f \colon X \to Y$ and $g \colon U \to V$, we say that f has the *left lifting property* (LLP) *with respect to* g and that g has the *right lifting property* (RLP) *with respect to* f if for any commutative square of solid arrows in $\mathscr{C}$

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & U \arrow[d, "g"] \\ Y \arrow[r] \arrow[ur, dotted] & V \end{tikzcd}$$

there exists a lift $Y \to U$ making the above diagram commuting in every direction.

We are now ready to state the definition of a model category.

Definition 1.1.2 ([Hov99, Definition 1.1.3])**.** A *model structure* on a category $\mathscr{C}$ is the datum of three classes of maps of $\mathscr{C}$, called *weak equivalences*, *fibrations*, and *cofibrations*, together with two functorial factorizations $\left(\mathrm{cF}^{\mathrm{triv}}, \mathrm{F}\right)$ and $\left(\mathrm{cF}, \mathrm{F}^{\mathrm{triv}}\right)$, satisfying the following axioms.

(A1) Weak equivalences satisfy the *two-out-of-three property*: given $f \colon X \to Y$ and $g \colon Y \to Z$ two maps in $\mathscr{C}$, then if two among f, g and $g \circ f$ are weak equivalences so is the third one.

(A2) If $f \colon X \to Y$ is a retract of $g \colon U \to V$ and g is a weak equivalence, a fibration, or a cofibration, so is f.

(A3) Let us define *trivial fibrations* as those fibrations that are also weak equivalences; analogously, let us define *trivial cofibrations*. Then fibrations have the right lifting property with respect to trivial cofibrations, and conversely cofibrations have the left lifting property with respect to trivial fibrations.

(A4) For any map $f \colon X \to Y$, $\mathrm{cF}^{\mathrm{triv}}(f)$ is a trivial cofibration, $\mathrm{F}(f)$ is a fibration, $\mathrm{cF}(f)$ is a cofibration and $\mathrm{F}^{\mathrm{triv}}(f)$ is a trivial fibration.

A *model category* is a category $\mathscr{C}$ admitting all limits and colimits and equipped with a model structure.

Remark 1.1.3. Definition 1.1.2 is *not* the only possible definition of model categories. For example, in his original paper [Qui67], Quillen did not ask for the existence of all limits and colimits, while Axiom 1.1.2.(A4) can be weakened as follows: one can only ask for two *not necessarily functorial* different factorizations for any morphism. For most of the examples we are interested in, however, factorizations indeed can be made functorial and our categories are both complete and cocomplete, so it is more or less harmless to include these requirements.

We are not really interested in delving deep in the theory of model categories. Thus, the following results are proposed as simple exercises to the reader.

Exercises 1.1.4.

(1) Prove that there are always *three* model structures on a complete and cocomplete category $\mathscr{C}$, defined by setting one among weak equivalences, fibrations or cofibrations to coincide with the class of isomorphisms, and setting the other two to be the class of *all* morphisms in $\mathscr{C}$.

(2) Prove that if $\mathscr{C}$ is equipped with a model structure, then $\mathscr{C}^{\mathrm{op}}$ admits a dual model structure: how is it defined?

(3) (Retract argument) Given a map $f: X \to Y$ in $\mathscr{C}$, admitting a factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, bend right, "i"'] & & Y \\ & A \arrow[ur, bend right, "p"'] & \end{tikzcd}$$

prove that if f has the left (resp. right) lifting property with respect to p (resp. with respect to i), then f is a retract of i (resp. of p).

(4) Choosing a class of weak equivalences and a class between fibrations and cofibrations uniquely determine the third one. Namely, in a model category cofibrations are precisely those maps enjoying the left lifting property with respect to all trivial fibrations, and trivial cofibrations are precisely those maps enjoying the left lifting property with respect to all fibrations, so one could simply ask these classes of morphisms to provide weak factorization systems. Try to deduce this statement, and its dual as well. (Hint: apply the retract argument of Exercise 1.1.4.(3) to some suitable factorization provided by Axiom 1.1.2.(A4).)

(5) Prove that (trivial) cofibrations are stable under pushouts, and that (trivial) fibrations are stable under pullbacks.

Definition 1.1.5. Given a model category $\mathscr{C}$, let us denote by $\mathbb{1}$ the terminal object, and by $\varnothing$ the initial object. We say that an object X is *fibrant* if the canonical map $X \to \mathbb{1}$ is a fibration; dually, we say that an object Y is *cofibrant* if the canonical map $\varnothing \to X$ is a cofibration.

Fibrant objects and cofibrant objects are naturally gathered in two distinct full subcategories of $\mathscr{C}$, that we shall denote by $\mathscr{C}_{\mathrm{f}}$ and $\mathscr{C}_{\mathrm{c}}$, respectively. The category of both cofibrant and fibrant objects is the intersection of these two categories, and is denoted by $\mathscr{C}_{\mathrm{cf}}$.

Remark 1.1.6. In virtue of Axiom 1.1.2.(A4), we can factor the initial morphism $\varnothing \to X$ as a cofibration $\varnothing \to QX$ followed by a trivial fibration $QX \to X$. Similarly, we can factor the terminal morphism $X \to \mathbb{1}$ as a trivial cofibration $X \to RX$ followed by a fibration $RX \to \mathbb{1}$. Moreover, the associations $X \mapsto QX$ and $X \mapsto RX$ define functors

$$Q\colon \mathscr{C} \longrightarrow \mathscr{C}_{\mathrm{c}}$$

and

$$R\colon \mathscr{C} \longrightarrow \mathscr{C}_{\mathrm{f}}.$$

We shall call Q and R the *cofibrant replacement* and the *fibrant replacement* functors, respectively.

1.2. **The homotopy category of a model category.**

Definition 1.2.1. Given a model category $\mathscr{C}$ with class of weak equivalences $\mathscr{W}$, the *homotopy category of* $\mathscr{C}$ is the localization $\mathrm{h}\mathscr{C} := \mathscr{C}\left[\mathscr{W}^{-1}\right]$. It is the *universal category* admitting a functor $\mathscr{C} \to \mathrm{h}\mathscr{C}$ sending all morphisms in $\mathscr{W}$ to isomorphisms.

There is a "standard" way to present the homotopy category of a model category $\mathscr{C}$. Namely, one can consider the same objects as $\mathscr{C}$ but arrows given in the following way: one adds formally an arrow $w^{-1}\colon Y \to X$ for any weak equivalence $w\colon X \to Y$, and then defines

$$\mathrm{Hom}_{\mathrm{h}\mathscr{C}}(X,Y) := \left\{ X \xrightarrow{f_1} X_1 \to \ldots \to X_{n-1} \xrightarrow{f_n} Y \,\middle|\, \begin{array}{c} \text{the arrows } f_i \text{ are either maps in } \mathscr{C} \\ \text{or formal inverses of maps in } \mathscr{W} \end{array} \right\}$$

imposing the relations

$$\left\{ X \xrightarrow{f} Y \xrightarrow{g} Z \right\} \simeq \left\{ X \xrightarrow{g\circ f} Z \right\}$$

for any composable maps f and g in $\mathscr{C}$, and

$$\left\{ X \xrightarrow{w^{-1}} Y \xrightarrow{w} X \right\} \simeq \left\{ X \xrightarrow{\mathrm{id}_X} X \right\}$$

for any weak equivalence w. However, there are some problems in general: if one takes *arbitrary* strings of maps the size can soon explode, see for example [Gin]. Indeed, if $\mathscr{C}$ is locally small but not small, there is no reason for the homotopy category defined above to be again locally small, and thus one *a priori* should be forced to move up some larger universe $\mathscr{U}$ where $\mathrm{h}\mathscr{C}$ becomes locally $\mathscr{U}$-small. But for model categories it turns out that this step is unnecessary.

Definition 1.2.2.

(1) A *cylinder object for* X is any object $\mathrm{Cyl}(X)$ such that the codiagonal map $\nabla\colon X \coprod X \to X$ factors through a cofibration $X \coprod X \to \mathrm{Cyl}(X)$ and a weak equivalence $\mathrm{Cyl}(X) \to X$. By precomposing with the natural inclusions $X \to X \coprod X$, one has maps ι_1 and ι_2 from X to $\mathrm{Cyl}(X)$.

(2) A *path object for* X is any object $\mathrm{Path}(X)$ such that the diagonal map $\Delta\colon X \to X \times X$ factors through a weak equivalence $X \to \mathrm{Path}(X)$ and a fibration $\mathrm{Path}(X) \to X \times X$. By postcomposing with the natural projections $X \times X \to X$, one has maps π_1 and π_2 from $\mathrm{Path}(X)$ to X.

(3) A *left homotopy* from $f\colon X \to Y$ to $g\colon X \to Y$ is a map $H\colon \mathrm{Cyl}(X) \to Y$ such that $H \circ \iota_1 = f$ and $H \circ \iota_2 = g$. We shall write $f \sim_l g$ if f is left homotopic to g.

(4) A *right homotopy* from $f\colon X \to Y$ to $g\colon X \to Y$ is a map $K\colon X \to \mathrm{Path}(Y)$ such that $\pi_1 \circ K = f$ and $\pi_2 \circ K = g$. We shall write $f \sim_r g$ if f is right homotopic to g.

(5) We say that $f\colon X \to Y$ and $g\colon X \to Y$ are *homotopic* ($f \sim g$) if they are both left and right homotopic.

(6) We say that $f\colon X \to Y$ is a *homotopy equivalence* if there exists $g\colon Y \to X$ such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$.

It turns out that these relations are *very* ill-behaved: in general, they not stable under compositions either to the left or to the right with other morphisms, they are not *equivalence* relations, and one does not imply the other. However, everything works neatly if the source is assumed to be cofibrant and the target is assumed to be fibrant: in particular, if we restrict ourselves to the category of both fibrant and cofibrant objects $\mathscr{C}_{\mathrm{cf}}$, all these issues disappear ([Hov99, Proposition 1.2.5, Corollaries 1.2.6 and 1.2.7]). Hence, one can define the "quotient" category $\mathscr{C}_{\mathrm{cf}/\sim}$, by taking the full subcategory of $\mathscr{C}$ spanned by both fibrant and cofibrant objects and then modding out the hom-sets by the homotopy equivalence relation. *This* turns out to be the correct definition of the homotopy category of $\mathscr{C}$.

Theorem 1.2.3 ([Hov99, Proposition 1.2.3, Corollary 1.2.9, Theorem 1.2.10]). *The natural map*

$$\mathrm{h}\mathscr{C}_{\mathrm{cf}} := \mathscr{C}_{\mathrm{cf}}\left[\mathscr{W}^{-1}\right] \longrightarrow \mathscr{C}_{\mathrm{cf}/\sim}$$

from the homotopy category of $\mathscr{C}_{\mathrm{cf}}$ to the naive homotopy category $\mathscr{C}_{\mathrm{cf}/\sim}$ is actually an equivalence of categories. Moreover, the natural inclusion $\mathscr{C}_{\mathrm{cf}} \hookrightarrow \mathscr{C}$ induces an equivalence of categories

$$\mathrm{h}\mathscr{C}_{\mathrm{cf}} \xrightarrow{\simeq} \mathrm{h}\mathscr{C}.$$

Remark 1.2.4. There is a certain amount of heuristics hidden in the statement of Theorem 1.2.3.

(1) The fact that $\mathrm{h}\mathscr{C}_{\mathrm{cf}} \simeq \mathscr{C}_{\mathrm{cf}/\sim}$ is due to the fact that fibrant-cofibrant objects of a model category $\mathscr{C}$ are precisely the ones for which holds a *Whitehead-like theorem*: a weak equivalence $X \to Y$ *always admits a quasi-inverse* $Y \to X$ providing an explicit homotopy equivalence between X and Y.

(2) The fact that $\mathscr{C}_{\mathrm{cf}/\sim} \simeq \mathrm{h}\mathscr{C}$ implies that we have a nice way to control the hom-sets in $\mathrm{h}\mathscr{C}$. Namely, given (the class of) two objects X and Y in $\mathscr{C}$, to compute $\mathrm{Hom}_{\mathrm{h}\mathscr{C}}(X, Y)$ it suffices to replace them with their cofibrant-fibrant replacement $QR(X)$ and $QR(Y)$ and them compute morphisms between them up to the equivalence relation of homotopy equivalence. Actually, there is more: it suffices to replace X by a cofibrant replacement QX and Y by a fibrant replacement RY.

(3) In his original book [Qui67], Quillen explicitly warns the reader that "model category" is meant to be interpreted as a short cut to *category of models for a homotopy theory*. Indeed, the model structure on $\mathscr{C}$ is simply a collection of additional data useful to present and study its homotopy category $\mathrm{h}\mathscr{C}$, just as one studies smooth differential manifolds by choosing carefully an atlas of smooth charts, or studies a group by choosing a suitable presentation by generators and relations. In particular, *different model structures can present the same homotopy theory*: the latter depends *only* on the class of weak equivalences.

We conclude this section by introducing very briefly the concepts of *Quillen adjunctions*, *Quillen equivalences*, and *derived functors*.

Definition 1.2.5 ([Hov99, Definitions 1.3.1, 1.3.6, 1.3.12]). Let $\mathscr{C}$ and $\mathscr{D}$ be two model categories, and let $F_L : \mathscr{C} \rightleftarrows \mathscr{D} : F_R$ be an adjunction between them.

(1) The pair $F_L \dashv F_R$ is a *Quillen adjunction* if F_L and F_R satisfy one of the following equivalent conditions.
- F_L preserves cofibrations and trivial cofibrations.
- F_R preserves fibrations and trivial fibrations.
- F_L preserves cofibrations and F_R preserve fibrations.

- F_L preserves trivial cofibrations and F_R preserves trivial cofibrations.

  In this case, we say that F_L *is a left Quillen functor* and that F_R *is a right Quillen functor.*

(2) The *left derived functor* $\mathbb{L}F_L$ of the left Quillen functor F_L is given by the composition

$$\mathbb{L}F_L \colon \mathrm{h}\mathscr{C} \xrightarrow{\mathrm{h}Q} \mathrm{h}\mathscr{C}_{\mathrm{c}} \xrightarrow{\mathrm{h}F_L} \mathrm{h}\mathscr{D}.$$

Dually, the *right derived functor* $\mathbb{R}F_R$ of the right Quillen functor F_R is given by the composition

$$\mathbb{R}F_R \colon \mathrm{h}\mathscr{D} \xrightarrow{\mathrm{h}R} \mathrm{h}\mathscr{D}_{\mathrm{f}} \xrightarrow{\mathrm{h}F_R} \mathrm{h}\mathscr{C}.$$

(3) The adjunction $F_L \dashv F_R$ is a *Quillen equivalence* if $\mathbb{L}F_L$ and $\mathbb{R}F_R$ form an adjoint equivalence between $\mathrm{h}\mathscr{C}$ and $\mathrm{h}\mathscr{D}$.

Remark 1.2.6. If $F_L \dashv F_R$ satisfy the equivalent conditions of Definition 1.2.5.(1), then $\mathbb{L}F_L \colon \mathrm{h}\mathscr{C} \rightleftharpoons \mathrm{h}\mathscr{D} \colon \mathbb{R}F_R$ is indeed an adjunction between the homotopy categories.

Example 1.2.7. Two particularly interesting examples of derived functors are the *homotopy limit* and *homotopy colimit* functors. Let us start with two observations.

(1) For any category $\mathscr{C}$ admitting limits and colimits of shape I, there exist a limit functor

$$\lim \colon \mathrm{Fun}(I, \mathscr{C}) \longrightarrow \mathscr{C},$$

a colimit functor

$$\mathrm{colim} \colon \mathrm{Fun}(I, \mathscr{C}) \longrightarrow \mathscr{C}$$

and a constant functor

$$\mathrm{const} \colon \mathscr{C} \longrightarrow \mathrm{Fun}(I, \mathscr{C})$$

which are related by a string of adjunctions $\mathrm{colim} \dashv \mathrm{const} \dashv \lim$.

(2) Under some assumptions on the model category $\mathscr{C}$ (the model structure has to be generated by "treatable and small enough data", in some sense), the category of diagrams $\mathrm{Fun}(I, \mathscr{C})$, where I is a small category, admits two model structures: the *projective model structure* which detects fibrations point-wise, and the *injective model structure* which detects cofibrations point-wise. Both detect weak equivalences point-wise, hence there exists an equivalence of categories

$$\mathrm{hFun}(I, \mathscr{C})_{\mathrm{inj}} \simeq \mathrm{hFun}(I, \mathscr{C})_{\mathrm{proj}}.$$

Then, we have two Quillen adjunctions

$$\mathrm{colim} \colon \mathrm{Fun}(I, \mathscr{C})_{\mathrm{proj}} \rightleftharpoons \mathscr{C} \colon \mathrm{const}$$

and

$$\mathrm{const} \colon \mathscr{C} \rightleftharpoons \mathrm{Fun}(I, \mathscr{C})_{\mathrm{inj}} \colon \lim,$$

which respectively determine a *homotopy colimit functor*

$$\mathrm{hocolim} := \mathbb{L}\,\mathrm{colim} \colon \mathrm{hFun}(I, \mathscr{C})_{\mathrm{proj}} \longrightarrow \mathrm{h}\mathscr{C}$$

and a *homotopy limit functor*

$$\mathrm{holim} := \mathbb{R}\lim \colon \mathrm{hFun}(I, \mathscr{C})_{\mathrm{inj}} \longrightarrow \mathscr{C}.$$

1.3. **Notable examples of model categories.** In this section we include the basic description of model structures on important categories, *with no proofs*. Namely, we describe the class of weak equivalences and one between fibrations and cofibrations (in virtue of Exercise 1.1.4.(4)), we describe fibrant-cofibrant objects if possible, and we exhibit either a path or cylinder objects providing the homotopy equivalence relation on fibrant-cofibrant objects.

Topological spaces. Let us start with the fundamental and motivating example of all the theory: topological spaces ([Hov99, Section 2.4]).

Definition 1.3.1.

(1) Let (S^n, e_1) be the unit sphere of $\mathbb{R}^{n+1}$, pointed at the point $(1,0,\dots,0)$. The *n-th homotopy group of a pointed topological space* (X,x) is the group[1]

$$\pi_n(X,x) := \mathrm{Hom}_{\mathrm{Top}_*}((S^n,e_1),(X,x))/\sim .$$

(2) A continuous map of topological spaces $f : X \to Y$ of topological spaces is a *weak homotopy equivalence* if it induces isomorphisms $\pi_n(X,x) \to \pi_n(Y,f(x))$ for every $n \geqslant 0$ and every choice of a basepoint $x \in X$.

(3) Let D^n be the unit disk in $\mathbb{R}^n$. A continuous map of topological spaces $X \to Y$ is a *Serre fibration* if for any $n \geqslant 0$ and for any diagram of solid continuous maps

$$\begin{tikzcd} D^n \arrow[r] \arrow[d, hook, "{(\mathrm{id},0)}"'] & X \arrow[d] \\ D^n \times [0,1] \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

there exists a lift $D^n \times [0,1] \to X$ making the diagram commute in every direction.

Theorem 1.3.2 ([Hov99, Theorem 2.4.19])**.** *The category* Top *of topological spaces admits a model structure where weak equivalences and fibrations are the weak homotopy equivalences and the Serre fibrations of Definition 1.3.1. In this model category, every topological space is fibrant, while cofibrant topological spaces are retracts of cell complexes ([Hov99, Definition* 2.4.3*]).*
A cylinder object for a topological space X is the topological product $X \times [0,1]$. In particular, left homotopies are exactly ordinary homotopies between continuous maps.

Remark 1.3.3. The Quillen model structure of Theorem 1.3.2 presents the homotopy theory which identifies objects sharing the same homotopy type, but we should note that the "naive" homotopy category Ho(Top) is itself the homotopy category for some model category. Namely, it is the homotopy category for the *Strøm model structure* on *all* topological spaces: see [Str72].

Remark 1.3.4. Every CW complex is a fibrant-cofibrant object for the model category on Top of Theorem 1.3.2: this implies the classical Whitehead theorem, which states that every weak homotopy equivalence between CW complexes is a *strict* homotopy equivalence, can be seen as a corollary of Theorem 1.2.3 via Remark 1.2.4.(1). In particular, the full category $\mathrm{CW} \subseteq \mathrm{Top}$ spanned by CW complexes is a proper subcategory of $\mathrm{Top}_{\mathrm{cf}}$. It is a remarkable result of algebraic topology that this inclusion induces an equivalence at the level of localizations at homotopy

[1] Recall that for $n = 0$ this is *a priori* just a pointed set.

equivalences: indeed, *every topological space* has the homotopy type of a CW complex, in virtue of [Hat02, Proposition 4.13]. In particular,

$$\mathrm{CW}\left[\mathscr{W}^{-1}\right] \simeq \mathrm{CW}_{/\sim} \simeq \mathrm{hTop}.$$

Simplicial sets. Even if not commonly studied here in Italy, the category of simplicial sets has to be cconsidered as the most important model category among all possible examples. And the reason is the following: we will see in Theorem 1.4.3 that their homotopy category presents the homotopy category of topological spaces, but the categorical properties of simplicial sets are way nicer and "more minimal".

Definition 1.3.5. The *simplex category* Δ is the category described as follows.

(1) Objects are non-empty finite ordinals $[n] := \{0 \to 1 \to \ldots \to n-1\}$.
(2) The maps in the hom-set $\mathrm{Hom}_\Delta([n],[m])$ are non-strictly order preserving functions from $[n]$ to $[m]$.

A *simplicial object* in a category $\mathscr{C}$ is a functor $X_\bullet \colon \Delta^{\mathrm{op}} \to \mathscr{C}$. The *category of simplicial objects of* $\mathscr{C}$ is the category of diagrams

$$\mathrm{s}\mathscr{C} := \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathscr{C}).$$

Remark 1.3.6. Even if *a priori* the amount of data for a simplicial object in $\mathscr{C}$ is huge, one can prove that maps in Δ are generated by all possible compositions of the following type of maps

(1) *Face maps* of the form $\delta^n_i \colon [n-1] \hookrightarrow [n]$, for $0 \leqslant i \leqslant n-1$, described as the unique injective map which skips the element $\{i\}$ in $[n]$.
(2) *Degeneracy maps* of the form $\sigma^n_i \colon [n+1] \twoheadrightarrow [n]$ for $0 \leqslant i \leqslant n$, described as the unique surjective map which sends both $\{i\}$ and $\{i+1\}$ to $\{i\}$.

These maps have to satisfy the *simplicial identities*.

(1) For $0 \leqslant i < j \leqslant n$, one has

$$\delta^{n+1}_i \circ \delta^n_j = \delta^{n+1}_{j+1} \circ \delta^n_i$$

and

$$\sigma^n_j \circ \sigma^{n+1}_i = \sigma^n_i \circ \sigma^{n+1}_{j+1}.$$

(2) For any $0 \leqslant i, j \leqslant n$, one has

$$\sigma^n_i \circ \delta^{n+1}_j = \begin{cases} \delta^n_i \circ \sigma^{n-1}_{j-1} & \text{for } i < j \\ \mathrm{id}_{[n]} & \text{for } i = j \text{ or } i = j+1\,. \\ \delta^n_{i-1} \circ \sigma^{n-1}_j & \text{for } i > j \end{cases}$$

In particular, a simplicial object $X_\bullet$ is equivalently described as

(1) A collection of objects X_n of $\mathscr{C}$ for any non-negative integer n.
(2) A collection of *face maps* $\mathrm{d}^n_i \colon X_n \to X_{n-1}$ and of *degeneracy maps* $\mathrm{s}^n_i \colon X_n \to X_{n+1}$ satisfying the dual of the simplicial identities.

Definition 1.3.7. The *category of simplicial sets* sSet is the category of simplicial objects in the category of sets.

Simplicial sets form a *topos* (indeed, by definition, they are gathered in a category of presheaves); hence the category of simplicial sets shares many interesting properties with the category of sets. For instance, sSet is a *Cartesian closed monoidal category*: the level-wise Cartesian product of simplicial sets provides a symmetric monoidal structure, and for any simplicial set $X_\bullet$ the functor

$$- \times X_\bullet : \mathrm{sSet} \longrightarrow \mathrm{sSet}$$

admits a right adjoint provided by the *mapping complex*

$$\underline{\mathrm{Map}}^\bullet_{\mathrm{sSet}}(X_\bullet, -) : \mathrm{sSet} \longrightarrow \mathrm{sSet}$$

defined on an object $Y_\bullet$ by the formula

$$\underline{\mathrm{Map}}^n_{\mathrm{sSet}}(X_\bullet, Y_\bullet) \coloneqq \mathrm{Hom}_{\mathrm{sSet}}(X_\bullet \times \Delta^n, Y_\bullet)$$

where Δ^n is the simplicial set corresponding to the representable functor $\mathrm{Hom}_\Delta(-, [n])$. In particular, we can consider sSet as a category *enriched over itself*, i.e., we have a *simplicial set* of maps between simplicial sets, and the composition is compatible with the Cartesian monoidal structure on sSet. The latter statement, for example, *does not hold* in Top: the product with a topological space X does not commute with colimits if X does not enjoy some nice properties (e.g., if X is locally compact Hausdorff).

Definition 1.3.8 ([Qui67, Chapter II, Section 3])**.** Let $f_\bullet : X_\bullet \to Y_\bullet$ be a map of simplicial sets.

(1) We say that $f_\bullet$ is a cofibration if it is a monomorphism in sSet (i.e., f_n is an injective map for any $n \geqslant 0$).

(2) We say that $f_\bullet$ is a fibration if it is a *Kan fibration*, i.e., if it has the right lifting property with respect to all inclusions $\Lambda^n_k \hookrightarrow \Delta^n$. Here, Λ^n_k is the sub simplicial set of Δ^n given by the union of the image of all face maps $s^n_j : \Delta^{n-1} \hookrightarrow \Delta^n$, except for the one corresponding to $j = k$.

(3) We say that $f_\bullet$ is a *trivial cofibration* (resp. *trivial fibration*) if it has the left lifting property with respect to all Kan fibrations (resp. if it has the right lifting property with respect to all monomorphisms).

(4) We say that $f_\bullet$ is a weak equivalence if it can be factorized as a composition $f_\bullet = p_\bullet \circ i_\bullet$, where $p_\bullet$ is a trivial fibration and $i_\bullet$ is a trivial cofibration.

Theorem 1.3.9 ([Qui67, Chapter II, Section 3, Theorem 3])**.** *The category* sSet *of simplicial sets admits a model structure where weak equivalences, fibrations and cofibrations are defined as in Definition 1.3.8. In this model category, every simplicial set is cofibrant, while fibrant simplicial sets are called Kan complexes.*
A cylinder object for a simplicial set $X_\bullet$ is the product of simplicial sets $X_\bullet \times \Delta^1$. Moreover, such model structure on simplicial sets is compatible with the simplicial enrichment and the monoidal structure, in the sense of [Qui67, Chapter II, Section 2, Definition 2].

Remark 1.3.10. It is necessary to provide some insight on the model structure described in Theorem 1.3.9.

(1) We did not describe weak equivalences for the model structure on the category of simplicial sets, and actually this is something that works in general: to characterize a

model structure it is sufficient to specify the classes of fibrations and cofibrations, while the rest can be deduced from lifting properties and from the existence of factorizations of morphisms.

(2) In this case, it turns out that weak equivalences of simplicial sets are those morphisms with right lifting property with respect to any inclusion $\partial\Delta^n \hookrightarrow \Delta^n$ (where $\partial\Delta^n$ is the sub simplicial set obtained by discarding the unique non degenerate n-simplex of Δ^n), or equivalently those morphisms of simplicial sets $f_\bullet : X_\bullet \to Y_\bullet$ such that for any Kan complex $K_\bullet$ precomposition of $f_\bullet$ induces a *simplicial homotopy equivalence*

$$\underline{\mathrm{Map}}^\bullet_{\mathrm{sSet}}(Y_\bullet, K_\bullet) \longrightarrow \underline{\mathrm{Map}}^\bullet_{\mathrm{sSet}}(X_\bullet, K_\bullet).$$

Most importantly, they are those morphisms whose geometric realization $|f_\bullet|$ is a weak homotopy equivalence in Top: this is actually how Hovey introduces them ([Hov99, Definition 3.2.1]). Here, we favored Quillen approach because it makes it more independent from Top, and because it makes it way less tautological the content of Theorem 1.4.3.

(3) The compatibility with the simplicial enrichment mentioned briefly at the end of Theorem 1.3.9 depends on some compatibility between the functors $-\times X_\bullet$ and $\underline{\mathrm{Map}}^\bullet_{\mathrm{sSet}}$ and (trivial) fibrations and (trivial) cofibrations. These are technical conditions but they produce the following statement, which will be made more rigorous in the following lectures: *the homotopy category* hsSet *of* sSet *is weakly enriched over itself*. Together with Theorem 1.4.3, this will provide an enrichment of hsSet over the homotopy category of topological spaces.

Chain complexes of modules. Finally, let us focus on the main algebraic example of a model category: namely, the category of chain complexes of (either left or right) R-modules $\mathrm{C}_\bullet(R)$ over an associative ring R. Actually, we shall describe a *projective model structure* on the category of non-negatively graded chain complexes $\mathrm{C}_{\geqslant 0}(R)$, an *injective model structure* on the category of non-positively graded chain complexes $\mathrm{C}_{\leqslant 0}(R)$, and a projective model structures on the category of *all* chain complexes $\mathrm{C}_\bullet(R)$. One can find in [Wei94] all relevant (and basic) definitions for chain complexes: we just remark that we proudly adopt a *homological* convention. (In particular, we interpret *bounded below cochain complexes* in terms of *bounded above chain complexes*.)

Theorem 1.3.11.

(1) ([Hov99, Theorem 2.3.11]) The category $\mathrm{C}_{\geqslant 0}(R)$ of non-negatively graded chain complexes of left R-modules admits a projective model structure where weak equivalences are quasi-isomorphisms and fibrations are surjections in every positive degree. Every object is fibrant, and cofibrant objects are those complexes which are projective in every non-negative degree.

(2) ([Hov99, Theorem 2.3.13]) The category $\mathrm{C}_{\leqslant 0}(R)$ of non-positively graded chain complexes of left R-modules admits an injective model structure where weak equivalences are quasi-isomorphisms and cofibrations are injections in every negative degree. Every object is cofibrant, and fibrant objects are those complexes which are injective in every non-negative degree.

(3) ([BMR14, Theorem 1.4]) The category $\mathrm{C}_\bullet(R)$ of all chain complexes of left R-modules admits a projective model structure where weak equivalences are quasi-isomorphisms and fibrations are surjections in every degree. Every object is cofibrant and fibrant objects

are chain complexes $C_\bullet$ which are projective in each degree and such that for any acylic object $A_\bullet$ the internal mapping chain complex $\underline{\mathrm{Map}}_R(C_\bullet, A_\bullet)$ *is again acyclic ([BMR14, Proposition* 1.7*]).*

Remark 1.3.12. A cylinder object for the model structures on chain complexes of Theorem 1.3.11 is not easy to describe in full generality. If R is commutative, there is a standard way to produce an *interval object* in (bounded below) chain complexes of left R-modules via the two-term complex

$$I_\bullet := \left[\dots \longrightarrow \underset{\deg=1}{R} \xrightarrow{\langle \mathrm{id}, -\mathrm{id}\rangle} \underset{\deg=0}{R \oplus R} \right],$$

and so by mimicking the construction of the cylinder object in sSet and Top one could define

$$\mathrm{Cyl}(C_\bullet) := C_\bullet \otimes_R I_\bullet$$

Yet, if $C_\bullet$ is not projective in each degree, the standard map

$$C_\bullet \oplus C_\bullet \longrightarrow C_\bullet \otimes_R I_\bullet$$

is *not* a cofibration for the projecive model structure. Anyway, this is a false problem: homotopies in these model categories are precisely chain homotopies.

1.4. **Relations between some of these model categories.** We conclude this section with a brief survey on how these model categories interact and are, in some sense, very different presentations of very similar and connected homotopy theories. Some of the well-known results of this theory will provide further foundational motivation for introducing ∞-categories in Section 2. Quite remarkably, the following discussion stems – *again* – as a non-trivial consequence of Yoneda lemma.

Theorem 1.4.1 (Ninja Yoneda Lemma). *The Yoneda embedding*

$$よ\colon \mathscr{C} \hookrightarrow \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathrm{Set})$$

is a fully faithful functor and exhibits $\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathrm{Set})$ *as the free cocompletion of the category* $\mathscr{C}$*, i.e., for any functor* $F\colon \mathscr{C} \to \mathscr{D}$ *with cocomplete target there exists an essentially unique realization functor*

$$|-|\colon \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathrm{Set}) \longrightarrow \mathscr{D},$$

which commutes with colimits and makes the following diagram commute.

$$\begin{tikzcd} \mathscr{C} \arrow[rr, hook, "よ"] \arrow[dr, bend right, "F"'] & & \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathrm{Set}) \arrow[dl, bend left, "|-|"] \\ & \mathscr{D} & \end{tikzcd}$$

Moreover, $|-|$ *is a left adjoint to the nerve functor* $\mathrm{N}\colon \mathscr{D} \to \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathrm{Set})$ *defined on an object D of $\mathscr{D}$ as the presheaf*

$$\mathrm{N}(D)\colon C \mapsto \mathrm{Hom}_{\mathscr{D}}(F(C), D).$$

Remark 1.4.2. The tongue-in-cheek name of Theorem 1.4.1 is shamelessly borrowed from [Lor21, Proposition 2.1]: this is how the classical statement *every presheaf on a category $\mathscr{C}$ is* *canonically a colimit of all the representable presheaves mapping to it* (or at least, its reformulation in terms of coend calculus) is referred to. This result is also known as *Density Formula* or *co-Yoneda Lemma* (in nLab, [nLa23a]): a proof of it can be found in [Mac71, Chapter III, §7, Theorem 1]. It is however an easy exercise to show how *that* Ninja Yoneda Lemma implies our Theorem 1.4.1: a proof can be found in the brilliant paper [Dug09, Proposition 2.2.4].

Theorem 1.4.1 has remarkable consequences. What happens if $\mathscr{C} = \Delta$? In this case, the category of presheaves on Δ is by definition the category of simplicial sets sSet. In the following, we shall study what happens when applying Theorem 1.4.1 to the simplex category, changing suitably the target $\mathscr{D}$.

Homotopy theory and simplicial sets. Let us start with $\mathscr{D} = \mathrm{Top}$. We can see that there exists a familiar functor from Δ, given by sending an object $[n]$ to the *standard topological n-simplex*

$$\Delta^n := \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=0}^{n} x_i = 1 \right\}.$$

Since Top is cocomplete, Theorem 1.4.1 provides a *geometric realization functor* $|{-}| : \mathrm{sSet} \to \mathrm{Top}$, together with a right adjoint. But in this case, such right adjoint bears a familiar face: it is the *singular simplicial complex* functor Sing. Moreover, it turns out that this adjunction behaves particularly well with respect to both model structures on sSet and Top.

Theorem 1.4.3 ([May67, Chapter III, §16])**.** *The adjunction*

$$|{-}| : \mathrm{sSet} \rightleftarrows \mathrm{Top} : \mathrm{Sing}$$

is a Quillen equivalence.

Let us delve a bit deeper into the meaning of Theorem 1.4.3.

(1) Simplicial sets contain *all* the homotopical information of topological spaces; yet, the categorical and abstract properties of the category sSet are *way* nicer.
(2) Homotopy groups of Kan complexes (which, being the fibrant-cofibrant objects of sSet, are essentially the only simplicial sets that matter for investigating the homotopy theory of topological spaces) are not as familiar as homotopy groups of topological spaces, but this is not a real problem: one can compute $\pi_n(S_\bullet, s_0)$ in terms of $\pi_n(|S_\bullet|, |s_0|)$ – actually they can be *defined* in this way.
(3) Most importantly, simplicial sets can be studied by combinatorial means, making them more tractable than topological spaces.

Category theory and simplicial sets. Let us change now the target from Top to Cat, the category of small[2] categories. Again, we have a standard way to embed Δ inside Cat, by sending $[n]$ to its categorical incarnation as a poset. This provides a functor

$$\mathrm{h} : \mathrm{sSet} \longrightarrow \mathrm{Cat}, \tag{1.4.4}$$

[2]This could appear as an underwhelming deal breaker of the theory. Fortunately, most of the time we can overcome any limitation on the size of our considered categories thanks to an unscrupulous, yet standard, abuse of the assumption that we are provided a sufficiently vast supply of Grothendieck universes, so as to enlarge our universe – hence, assume our categories to be small – whenever we need to.

that we call (for reasons that will be more apparent in Section 2) the *homotopy category* functor. The image of a simplicial set $S_\bullet$ under h is the category described as follows.

(1) Objects are given by the vertices of $S_\bullet$.
(2) The hom-sets $\mathrm{Hom}_{\mathrm{h}S_\bullet}(x, y)$ are generated by edges $f \in S_1$ such that $\mathrm{d}_0(f) = x$ and $\mathrm{d}_1(f) = y$, modulo the relations $\mathrm{s}_0(x) = \mathrm{id}_x$ for any $x \in S_0$ and $\mathrm{d}_1(\sigma) = \mathrm{d}_0(\sigma) \circ \mathrm{d}_1(\sigma)$ for any $\sigma \in S_2$.

However, we are actually more interested in its right adjoint $\mathrm{N} \colon \mathrm{Cat} \to \mathrm{sSet}$. Reading the statement of Theorem 1.4.1, we can see that N is the functor sending a category $\mathscr{C}$ to the simplicial set $\mathrm{N}\mathscr{C}$ whose set of n-simplices is $\mathrm{Hom}_{\mathrm{Cat}}([n], \mathscr{C})$, i.e.:

(1) The set of 0-simplices is the set of objects of $\mathscr{C}$.
(2) The set of n-simplices is the set of strings of composition of n compatible maps.
(3) The degeneracy maps $\mathrm{d}_i^n \colon \mathrm{N}\mathscr{C}_n \to \mathrm{N}\mathscr{C}_{n-1}$ collapse an n-simplex, corresponding to a composition $X_0 \to \ldots \to X_i \to X_{i+1} \to X_{i+2} \to \ldots \to X_n$, onto the composition $X_0 \to \ldots \to X_i \to X_{i+2} \to \ldots \to X_n$.
(4) The face maps $\mathrm{s}_i^n \colon \mathrm{N}\mathscr{C}_{n-1} \to \mathrm{N}\mathscr{C}_n$ insert the identity on the object X_i into a composition $X_0 \to \ldots \to X_{n-1}$.

Theorem 1.4.5 ([Rez21, Proposition 4.10])**.** *The nerve functor* $\mathrm{N} \colon \mathrm{Cat} \to \mathrm{sSet}$ *is fully faithful.*

So, up to size issues, even ordinary category theory can be fully recovered inside the theory of simplicial sets.

Chain complexes and simplicial abelian groups. Finally, let us review the connection between chain complexes of $\mathbb{Z}$-modules and simplicial abelian groups. This is a consequence of some *enriched version* of Theorem 1.4.1, see for instance [Kan58].

Definition 1.4.6. A *simplicial abelian group* is a simplicial object in the category of abelian groups Ab. The category of simplicial abelian groups is denoted by sAb.

Theorem 1.4.7 (Dold-Kan correspondence, [Dol58; Kan58; DP61])**.** *There is an equivalence of categories*

$$\Gamma \colon \mathrm{C}_{\geqslant 0}(\mathrm{Ab}) \rightleftarrows \mathrm{sAb} \colon \mathrm{N},$$

where the functor N *is the normalized chain complex functor, sending a simplicial abelian group* $A_\bullet$ *to the chain complex described as follows.*

(1) In homological degree n,

$$\mathrm{N}(A_\bullet)_n := \bigcap_{i \geqslant 1} \ker \mathrm{d}_i^n.$$

(2) The differential $\mathrm{N}(A_\bullet)_n \to \mathrm{N}(A_\bullet)_{n-1}$ *is the degeneracy map* d_0^n*.*

Theorem 1.4.7 was later refined as follows. In [Qui67, Chapter II, Section 4, Theorem 4], Quillen proved that the category sAb admits a natural model structure for which every object is fibrant: using the formalism of *transferred model structures* developed in [Cra95, Theorem 3.3], this model structure can be characterized by the fact that weak equivalences and fibrations are detected along the forgetful functor $\mathrm{sAb} \to \mathrm{sSet}$, while cofibrations are generated by the image of cofibrations in sSet under the free simplicial abelian group functor $\mathbb{Z}[-] \colon \mathrm{sSet} \to \mathrm{sAb}$. Then:

Theorem 1.4.8 ([Qui67, Chapter II, Section 4, Remark 5]). *The equivalence of categories of Theorem 1.4.7 is Quillen.*

In particular, even homological algebra can be recovered from the homotopy theory of simplicial sets, which is the same as the homotopy theory of topological spaces in virtue of Theorem 1.4.3. The key observation that the homotopy theory of simplicial sets is ubiquitous and retains all information coming from algebraic topology, homological algebra *and* category theory is the stepping stone for the theory of ∞-categories that we shall outline in Section 2.

1.5. **Some exercises.**

(1) Try to prove some of the statements of this section. If you are an algebraic geometer with a penchant for homological algebra, it should be sufficiently easy to prove in all details the existence of the projective model structure on bounded below chain complexes of left R-modules, at least up to reading the statements of some technical results in [Hov99] as "hints".

(2) (To read the full story hinted at by this exercise – i.e., that model categories with a reasonably compatible closed monoidal structure produce closed monoidal homotopy categories – read [Hov99, Chapter 4].) Prove that if $P_\bullet$ is a complex of projectives, then the adjunction

$$- \otimes_R P_\bullet \colon \mathrm{C}_\bullet(R) \rightleftarrows \mathrm{C}_\bullet(R) \colon \underline{\mathrm{Hom}}_R(C_\bullet, -)$$

yields a Quillen adjunction

$$- \otimes_R P_\bullet \colon \mathrm{C}_\bullet(R)_{\mathrm{proj}} \rightleftarrows \mathrm{C}_\bullet(R)_{\mathrm{proj}} \colon \underline{\mathrm{Hom}}_R(C_\bullet, -).$$

Show that if $P_\bullet \to M[0]$ is a quasi-isomorphism, the derived functors $\mathbb{L}(- \otimes_R P_\bullet)$ and $\mathbb{R}\underline{\mathrm{Hom}}_R(P_\bullet, -)$ in the sense of Definition 1.2.5 agree with the usual definition of derived tensor product $- \otimes_R^{\mathbb{L}} M$ and derived hom-functor $\mathbb{R}\mathrm{Hom}_R(M, -)$. (Note that the derived hom-functor is usually presented via an *injective* resolution of the target. The fact that one can equivalently resolve the source by a complex of projectives is classically a standard yoga exercise in spectral sequence theory: here, it is a simple consequence of the fact that the homotopy categories underlying $C_\bullet(R)_{\mathrm{proj}}$ and $C_\bullet(R)_{\mathrm{inj}}$ coincide.)

(3) (For the following, use the so-called *Quillen formula* to detect homotopy pullbacks and pushouts via strict pullbacks and pushouts, see for instance [Lur09, Proposition A.2.4.4]) Compute the homotopy colimit of the diagram $\{*\} \leftarrow \{*\} \coprod \{*\} \to \{*\}$ in Top. Try to visualize the geometric picture.

(4) Compute the homotopy colimit of the diagram $0 \leftarrow M_\bullet \to 0$ in $\mathrm{C}_\bullet(R)$. Then compute the homotopy limit of the diagram $0 \to M_\bullet[1] \leftarrow 0$. Deduce some properties of homotopy limits and colimits in the category of chain complexes.

(5) Prove that the singular complex of a topological space is always a Kan complex.

(6) Prove that the nerve of a small category $\mathscr{C}$ enjoys the following property: for any *inner horn* $\Lambda^n_k \subseteq \Delta^n$, with $0 < k < n$, and for any map $\Lambda^n_k \to \mathrm{N}\mathscr{C}$, there exists a *unique* lift

making the following diagram commute.

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & \mathrm{N}\mathscr{C} \\ \Delta^n \arrow[ur, bend right] & \end{tikzcd}$$

There is more: prove that any simplicial set $S_\bullet$ satisfying this property arises as the nerve of a small category $\mathscr{C}$. Why did we restrict ourselves to *inner* horns? Can you imagine an outer horn which does not admit a filling in (the nerve of) a category $\mathscr{C}$?

(7) Prove that the nerve of a small category $\mathscr{C}$ is a Kan complex if and only if it is a small groupoid. There is more: any 2-*coskeletal* Kan complex admitting a unique filler for any horn inclusion $\Lambda^n_k \subseteq \Delta^n$ for $n \geqslant 2$ is the nerve of a groupoid. If you feel like it, you can try to prove it by yourself up to reading the definition of *skeletal* and *coskeletal* simplicial sets, see for instance [May67, Chapter II, §8].

(8) Try to investigate what happens to a topological space X under the composition

$$\Pi_1 \colon \mathrm{Top} \xrightarrow{\mathrm{Sing}} \mathrm{sSet} \xrightarrow{\mathrm{h}} \mathrm{Cat}.$$

What functor is this? What are the endomorphisms of an object x of $\Pi_1(X)$? What happens to the induced functors $\Pi_1(f)\colon \Pi_1(X) \to \Pi_1(Y)$ if f is a weak homotopy equivalence? What can we say about $\Pi_1(X)$ if X is 1-truncated (i.e., $\pi_n(X,x) \cong 0$ for any base point x and any $n \geqslant 2$)?

(9) The notion of derived functors of Definition 1.2.5.(2) does not assume our categories to be abelian: this allows for a theory of derived functors that transcends abelian categories with enough projectives and injectives. Test the power of such theory with the following example.

a. Using [Cra95, Theorem 3.3], prove (or convince yourself) that the category of simplicial commutative rings sCRing admits a model structure transferred by the one on simplicial abelian groups via the forgetful-free adjunction

$$\mathrm{sAb} \rightleftarrows \mathrm{sCRing}.$$

In particular, weak equivalences are weak equivalences of simplicial sets (i.e., quasi-isomorphisms of the corresponding normalized chain complexes) and fibrations are Kan fibrations (i.e., morphisms which are surjective in positive homological degrees).

b. Prove (or convince yourself) that the same claim holds for the category of B-augmented simplicial commutative rings $\mathrm{sCRing}_{/B}$. Here fibrations, cofibrations and weak equivalences are detected via the forgetful functor $\mathrm{sCRing}_{/B} \to \mathrm{sCRing}$.

c. The category $\mathrm{sCRing}_{/B}$ admits products, hence we can consider the category of *abelian group objects* $\mathrm{Ab}\big(\mathrm{sCRing}_{/B}\big)$. Prove that there exists an equivalence of categories

$$\mathrm{Ab}\big(\mathrm{sCRing}_{/B}\big) \simeq \mathrm{sMod}_B$$

provided by sending $\left\{A_\bullet \xrightarrow{\varepsilon} B\right\} \mapsto \ker(\varepsilon)$, and with inverse provided by the assignation $M_\bullet \mapsto B \oplus M_\bullet$ with level-wise square-zero commutative ring structure (for the non-simplicial statement, see for example [Qui70, Proposition 1.5]).

d. Prove that the forgetful functor

$$\mathrm{oblv}_{\mathrm{Ab}} \colon \mathrm{Ab}\big(\mathrm{sCRing}_{/B}\big) \simeq \mathrm{sMod}_B \longrightarrow \mathrm{sCRing}_{/B}$$

is a right Quillen functor, with left adjoint provided by the level-wise *Kähler differential functor*

$$\Omega^1_{-/B} \colon \mathrm{sCRing}_{/B} \longrightarrow \mathrm{sMod}_B.$$

e. Finally derive to the left this functor to obtain the functor

$$\mathbb{L}_{-/B} \colon \mathrm{hsCRing}_{/B} \longrightarrow \mathrm{hsMod}_B \simeq \mathrm{hC}_\bullet(B).$$

Congratulations: you just constructed the *cotangent complex* of a simplicial commutative ring at its given B-point ([Ill71; Ill72]).

(10) In virtue of the previous exercise, one could think: simplicial commutative rings are commutative monoids with respect to the symmetric monoidal structure on simplicial abelian groups given by level-wise tensor product. Since chain complexes are a nice symmetric monoidal category, why can't we repeat *verbatim* the previous arguments in the context of commutative monoids in chain complexes, i.e., *commutative differential-graded rings*? The answer depends on the fact that the Dold-Kan correspondence of Theorem 1.4.7 is *not* monoidal. The solution to this exercise provides an example of what can go wrong: prove that, if one works over a commutative ring R which *does not contain the field of rational numbers* $\mathbb{Q}$, the adjunction

$$\mathrm{Free}_{\mathrm{CAlg}} \colon \mathrm{C}_{\geqslant 0}(R) \rightleftarrows \mathrm{cdga}_R \colon \mathrm{oblv}_{\mathrm{CAlg}}$$

does not allow to lift the model structure on $\mathrm{C}_{\geqslant 0}(R)$ to cdga_R. (See, for instance, this answer at MathOverflow: [Tyl].)

(11) Prove the *simplicial Ninja Yoneda Lemma*: for any category $\mathscr{C}$, the category of *simplicial presheaves* $\mathrm{Fun}(\mathscr{C}, \mathrm{sSet})$ is the *universal model category generated by* $\mathscr{C}$, and it is the category which freely adds *homotopy colimits* to $\mathscr{C}$. Alternatively, read the proof of this powerful statement in [Dug01].

2. Definitions and basic constructions in ∞-category theory

Model categories were a first, quite powerful indeed, tool to study homotopical and homological phenomena via strict models. However, we have already seen the shortcomings of such tool: an obvious drawback, for example, is given by the fact we always need to switch objects and arrows with fibrant and cofibrant replacements. But there are also other annoying technical issues: for any model category $\mathscr{C}$ and for an arbitrary shape K, the categories $\mathrm{hFun}(K, \mathscr{C})$ and $\mathrm{Fun}(K, \mathrm{h}\mathscr{C})$ are usually – almost always – quite different; moreover, model structures often *fail* to fully capture the richness and abundance of structures that their homotopy categories are equipped with. So, we want to work in a *fully homotopical* (or *derived*) world, in which everything is naturally expressed up to homotopy: in order to do so, we need to *keep track* of the homotopies that testify that some objects are equivalent. Hopefully, in such world one can apply ordinary categorical tools (or suitable generalizations) without worrying about explicit models and strict isomorphisms.

Before proceeding in describing what this kind of "fully homotopical" world is, let us look for a moment back to Section 1, and let us summarize briefly what we have learned up to this moment, *including* statements deduced from the exercises of Section 1.5.

(1) Simplicial sets encode all the homotopical information of topological spaces.
(2) Category theory, up to size issues, is embedded inside the theory of simplicial sets.
(3) Homological algebra is in some sense a $\mathbb{Z}$-linear analogue of homotopy theory.
(4) The *fundamental groupoid* functor

$$\Pi_1 : \mathrm{Top} \xrightarrow{\mathrm{Sing}} \mathrm{sSet} \xrightarrow{|-|} \mathrm{Gpd} \subseteq \mathrm{Cat}$$

recovers all the homotopical information of a topological space X in degrees $\leqslant 1$.

Thus, we want to develop a theory that recovers simultaneously the homotopy of simplicial sets (i.e., of Kan complexes), hence the homotopy of topological spaces; but we also want to enlarge category theory in order to produce a notion of *n-groupoids* which has recover the homotopical information of topological spaces in degrees $\leqslant n$. In particular, we want to go up to ∞ and produce a theory of ∞-*groupoids* which can present faithfully all topological spaces up to homotopy equivalence: this is precisely the goal of the *homotopy hypothesis*. Finally, we want this theory to be well-behaved enough to do *algebra* in it, and recover classical homological algebra inside this broader theory. This section wants to cover the first part of this program, mostly categorical in nature; the algebraic part will be developed in **??**.

References for this section. Almost everything presented in this section is a survey on what the author has learned by reading [Lur09] and [RV22]. The rest is comprised of errors and misunderstandings of his.

2.1. What an ∞-category should look like.

Given a topological space X, its fundamental groupoid $\Pi_1(X)$ is the groupoid having as many objects as the points in X, and in which the hom-sets $\mathrm{Hom}_{\Pi_1(X)}(x, y)$ recover the homotopy classes of paths from x to y inside X. But we can refine this notion as follows.

(1) Given two points x and y in X, corresponding to two objects in $\Pi_1(X)$, we can consider the set of paths $x \to y$ in X.
(2) Given two paths $\alpha, \beta \colon x \to y$, we can consider the set of *homotopies* between α and β.

In this way, the hom-set $\mathrm{Hom}_{\Pi_1(X)}(x, y)$ is actually an object of the category $\mathrm{hTop}_{\leqslant 1}$, i.e., the subcategory of hTop spanned by those topological spaces whose homotopy groups are trivial for $n \geqslant 2$ and for any choice of the base-point: using the terminology of algebraic topologists, this is the category of *homotopy* 1*-types*. In other words, we have allowed both *maps between objects* and *homotopies between maps*, which can be composed but only in a weak, homotopical fashion: in particular, $\Pi_1(X)$ is naturally enriched over the category of homotopy 1-types.

Definition 2.1.1 (Enrichment of a category, [Bor94, Definition 6.2.1])**.** A category $\mathscr{C}$ is said to be *enriched* over a monoidal category $\mathscr{V}^{\otimes}$ if $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is an object in $\mathscr{V}$, and the composition $\mathrm{Hom}_{\mathscr{C}}(X, Y) \otimes \mathrm{Hom}_{\mathscr{C}}(Y, Z) \to \mathrm{Hom}_{\mathscr{C}}(X, Z)$ is a map of $\mathscr{V}$ satisfying suitable associativity conditions.

In order to detect the homotopical information of *all* topological spaces, and not only of those with trivial higher homotopy groups, our theory of ∞-categories should hence be a theory of categories enriched over the category of *all* homotopy types hTop. This means that we should be able to consider a set of morphisms between objects, but also homotopies between morphisms, 2-homotopies between homotopies, and so forth, producing k-morphisms between $(k-1)$-morphisms for all positive integers k: this is what the "∞" in front of "∞-categories" stands for. Moreover, we want all these higher morphisms to be invertible (indeed, these higher morphisms are meant to be homotopy *equivalences*). This whole formalism may seem unnecessarily convoluted at first glance, but it is actually rather natural for two reasons.

(1) Consider the definition of the hom-set $\mathrm{Hom}_{\Pi_1(X)}(x, y)$, which is equivalently the quotient of the subset of $\mathrm{Hom}_{\mathrm{Top}}([0,1], X)$ consisting of those maps sending 0 to x and 1 to y under the homotopy equivalence relation. Composition is given by concatenation of paths: one usually takes two compatible paths $\alpha, \beta \colon [0,1] \to X$ and then defines the path $\beta \circledast \alpha \colon [0,1] \to X$ by "running at twice the speed along α", and then by "running at twice the speed along β", i.e.,
$$(\beta \circledast \alpha)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2} \\ \beta(2t-1) & \text{if } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$
But why *this* parametrization? The following would work out just as fine for our scope.
$$(\beta \circledast \alpha)^{\sim}(t) = \begin{cases} \alpha(3t) & \text{if } 0 \leqslant t \leqslant \frac{1}{3} \\ \beta\left(\frac{3t-1}{2}\right) & \text{if } \frac{1}{3} \leqslant t \leqslant 1. \end{cases}$$
The fact is that $\beta \circledast \alpha$ and $(\beta \circledast \alpha)^{\sim}$ are homotopic as well. And again, there are *multiple* possible homotopies testifying to it, which are again homotopic one to the other, and so on.
(2) Consider the category of small categories Cat. In an arbitrary category, *isomorphisms* are defined to be those maps $f \colon X \to Y$ admitting an inverse $f^{-1} \colon X \to Y$ such that compositions are *strictly* equal to the identity of X and Y, respectively. Yet, in category theory

we soon experience the fact that strict isomorphism of categories are too restrictive, and that the better behaved notion is the one of *equivalences* of categories. An equivalence of categories is the datum of two functors $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$ together with natural equivalences of functors $\sigma\colon \mathrm{id}_{\mathscr{C}} \Rightarrow G \circ F$ and $\tau\colon \mathrm{id}_{\mathscr{D}} \Rightarrow F \circ G$: so, we ask for the *datum* of some natural transformation testifying to *how* these compositions resemble the identity functors on $\mathscr{C}$ and $\mathscr{D}$. But what is the datum of such a natural transformation if not a choice of some *homotopy*? Long story short: we have *always* been taught to think of Cat as a $(2,1)$-category, and not as a mere (1-)category! (Actually, we have always thought of Cat as a $(2,2)$-*category*, because in category theory we are interested also in *non-invertible* natural transformations between functors.)

In virtue of the above heuristics, we have two immediate models for the theory of ∞-categories.

Definition 2.1.2 (Topologically enriched categories)**.** A *topologically enriched category* is a category $\mathscr{C}$ enriched over the category $\mathrm{Top}_{\mathrm{cg}}$ of compactly generated and weakly Hausdorff topological spaces. We shall denote the category of topologically enriched categories with topological functors by $\mathrm{Cat}_{\mathrm{Top}}$.

Remark 2.1.3.

(1) Since the natural functor
$$\mathrm{Top} \longrightarrow \mathrm{hTop}$$
preserves products, we can "shift" the enrichment of a topologically enriched category from a Top-enrichment to a hTop-enrichment (for a reference for this result, see for instance [Rie14, Lemma 3.4.3]). We shall not distinguish in notation between a topological category $\mathscr{C}$ and its associated hTop-enriched category $\mathscr{C}$: this is motivated by the fact that the former is simply an explicit model for the latter, which is the "real" ∞-category $\mathscr{C}$.

(2) The assumption on the topological spaces providing the enrichment of topologically enriched categories is motivated by the fact that the homotopy theory of compactly generated and weakly Hausdorff topological spaces already presents the homotopy theory of *all* topological spaces; yet, the restriction to $\mathrm{Top}_{\mathrm{cg}}$ simplifies a lot of the technical machinery that we need in order to develop the theory. For example, when restricted to $\mathrm{Top}_{\mathrm{cg}}$, the Cartesian monoidal structure of Top is indeed *closed*, i.e., there is a well-defined *mapping space* which provides a right adjoint to the functor taking the Cartesian product with a fixed topological spaces: this is not true anymore if one considers the whole Top.

The complexity of phenomena arising in the topological framework already suggest that if one is interested in doing homotopy theory it could be reasonable to discard topological spaces altogether and focus only on simplicial sets.

Definition 2.1.4 (Simplicially enriched categories, [Qui67])**.** A *simplicially enriched category* is a category $\mathscr{C}$ enriched over the category sSet of simplicial sets. We shall denote the category of simplicially enriched categories and simplicial functors by Cat_{Δ}.

Again as in Remark 2.1.3.(1), any simplicially enriched category $\mathscr{C}$ comes equipped with a mate hsSet-enriched category (that we again denote by $\mathscr{C}$), having the same class of objects and with enrichment provided by the natural product-preserving functor map

$$\mathrm{sSet} \longrightarrow \mathrm{hsSet}.$$

Definition 2.1.5. Let $\mathscr{C}$ and $\mathscr{D}$ be either two topologically enriched categories or two simplicially enriched categories. A topological or simplicial functor $F\colon \mathscr{C} \to \mathscr{D}$ is a *weak equivalence* if it is an equivalence of hTop-enriched (or equivalently hsSet-enriched) categories, i.e., if and only if the following two conditions hold.

(1) The functor of ordinary categories $\mathrm{h}F\colon \mathrm{h}\mathscr{C} \to \mathrm{h}\mathscr{D}$ is essentially surjective.
(2) The map
$$\mathrm{Map}_{\mathscr{C}}(X, Y) \longrightarrow \mathrm{Map}_{\mathscr{D}}(F(X), F(Y))$$
is a weak equivalence for the standard model structure on topological spaces or simplicial sets, respectively.

It is not difficult to prove, let alone to believe, that Theorem 1.4.3 implies that the homotopical theories underlying topologically enriched categories and simplicially enriched categories are the same: we can switch between the two by replacing a topological mapping space with its singular complex, via a functor
$$\mathrm{Sing}\colon \mathrm{Cat}_{\mathrm{Top}} \longrightarrow \mathrm{Cat}_{\Delta}$$
and conversely by replacing a simplicial mapping space with its geometric realization via a functor
$$|-|\colon \mathrm{Cat}_{\Delta} \longrightarrow \mathrm{Cat}_{\mathrm{Top}}.$$
Moreover, this theory encompasses obviously category theory (an ordinary category is a topologically enriched category with discrete mapping spaces between any pair of objects); it is less obvious, but still fairly believable, how it encompasses homotopy theory of topological spaces (this amounts to producing an ∞-*groupoid* associated to a topological spaces). So, an ∞-category should be more or less something along these lines; yet, the necessary technical machinery to make the theory work with these models is quite cumbersome.

(1) In both these models, one can consider the *homotopy* 1-*category* of a topologically/simplicially enriched category $\mathscr{C}$, defined as the category $\pi_0\mathscr{C}$ with same objects and with hom-sets
$$\mathrm{Hom}_{\pi_0\mathscr{C}}(X, Y) := \pi_0\mathrm{Map}_{\mathscr{C}}(X, Y).$$
It is clear that there should be some adjunction going on between the category of small categories Cat and the category of simplicially/topologically enriched categories Cat_{Δ} / $\mathrm{Cat}_{\mathrm{Top}}$, provided by some sort of functor
$$\pi_0\colon \mathrm{Cat}_{\mathrm{Top}} \rightleftharpoons \mathrm{Cat}_{\Delta} \longrightarrow \mathrm{Cat}. \tag{2.1.6}$$
Yet, in the topological case, the *strict* adjunction works fine only on those categories whose mapping spaces are locally path-connected ([Lur09, Remark 1.2.3.2]).

(2) One encounters a lot of technical issues in defining a topologically/simplicially enriched category of topological/simplicial functors between two topologically/simplicially enriched categories $\mathscr{C}$ and $\mathscr{D}$. Very roughly, the issues boil down to the fact that functors between ∞-categories should provide *homotopy coherent diagrams*, i.e., they should consist of the datum of a functor between homotopy categories $F\colon \mathrm{h}\mathscr{C} \to \mathrm{h}\mathscr{D}$ together with all the natural transformations that allow to lift F to a honest functor between $\mathscr{C}$ and $\mathscr{D}$; but for technical reasons, this expectations is not met. (More in detail: both topologically enriched categories and simplicially enriched categories are endowed with a model structure, but even if $\mathscr{C}$ is cofibrant, hence we have enough rigid functors $\mathscr{C} \to \mathscr{D}$ to represent all ∞-functors, the product $\mathscr{C} \times \Delta^1$ needs not to be cofibrant anymore – and rigid functors $\mathscr{C} \times \Delta^1 \to \mathscr{D}$ are precisely the ones that represent *natural transformations* between ∞-functors from $\mathscr{C}$ to $\mathscr{D}$. See [Lur09, Sections 1.2.6 and 1.2.7] for more information about this technical stuff.)

Rather, we try to pursue the intuition sketched by the exercises of Section 1.5: *topological spaces are Kan complexes*, and *categories are almost Kan complexes*.

Definition 2.1.7 (Weak Kan complexes, [BV73] or quasi-categories, [Joy02])**.** A *weak Kan complex* or *quasi-category* is a simplicial set $Q_\bullet$ satisfying the following lifting property: for every *inner* horn $\Lambda^n_k \subseteq \Delta^n$, with $0 < k < n$, and any map $\Lambda^n_k \to Q_\bullet$, there exists a lift

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & Q_\bullet \\ \Delta^n \arrow[ur, bend right] & \end{tikzcd}$$

making the diagram commute.

We shall denote the full subcategory of sSet spanned by quasi-categories by $\mathrm{sSet}_{\mathrm{Joyal}}$.

Remark 2.1.8.

(1) Note that, in order to get to Definition 2.1.7, we have relaxed simultaneously the properties of the singular complex of a topological space and of the nerve of a category: the former has the lifting property against *all* inclusions of horns, the latter has a *unique* lifting with respect to inclusions of inner horns.

(2) We shall think of a vertex $x \in \mathscr{C}_0$ as an object of $\mathscr{C}$, and of an edge $f \in \mathscr{C}_1$ as a morphism between the source vertex $\mathrm{d}^1_0(f)$ and the target vertex $\mathrm{d}^1_1(f)$. In particular, each vertex x comes equipped with a canonical edge whose both endpoints coincide with x – namely, the degenerate edge $\mathrm{s}^1_0(x)$, which acts as the identity of x.

(3) Given two vertices x and y in a quasi-category $\mathscr{C}$, we can define a whole mapping space $\mathrm{Map}^\bullet_{\mathscr{C}}(x,y)$, which is the simplicial set described in degree n by the formula set

$$\mathrm{Map}^n_{\mathscr{C}}(x,y) := \{\sigma\colon \Delta^n \to \mathscr{C} \mid \sigma|_{\Delta^{\{0,\dots,n-1\}}} = x \text{ and } \sigma|_{\Delta^n} = y\}$$

and with faces and degeneracies induced by the ones of $\mathscr{C}$. It is easily seen that this is a Kan complex, i.e., it produces the homotopy type of a topological space: this is proved in [Lur09, Proposition 1.2.2.3]. In particular, we are still retaining our (pretty robust) intuition of ∞-categories as "categories enriched over homotopy types".

(4) For a simplicial set $S_\bullet \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$, precomposition with the natural involution $\Delta \xrightarrow{\simeq} \Delta$ which fixes the objects of Δ and which sends a map $f : [n] \to [m]$ to the map

$$f^{\mathrm{op}} \colon i \mapsto m - f(n-i)$$

produces a new simplicial set $S_\bullet^{\mathrm{op}} \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$. In the case of a quasi-category $\mathscr{C}$ then $\mathscr{C}^{\mathrm{op}}$ is the *opposite quasi-category of* $\mathscr{C}$.

(5) We will say that a morphism $f : x \to y$ is an *equivalence* in $\mathscr{C}$ if there exists an edge $f^{-1} \colon y \to x$ and a pair of 2-simplices

$$\begin{tikzcd} & y \arrow[dr, "f^{-1}"] \arrow[d, phantom, "\Downarrow"] & \\ x \arrow[ur, "f"] \arrow[rr, equal, "\mathrm{id}_x"'] & & x \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & x \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow"] & \\ y \arrow[ur, "f^{-1}"] \arrow[rr, equal, "\mathrm{id}_y"'] & & y. \end{tikzcd}$$

Definition 2.1.9. Given two quasi-categories $\mathscr{C}$ and $\mathscr{D}$, a *quasi-functor* $F : \mathscr{C} \to \mathscr{D}$ is a map of simplicial sets.

Remark 2.1.10.

(1) In Definition 2.1.7, all the technical disadvantages that we encountered when trying to define topologically/simplicially enriched functors disappear, because defining the action of F on *all* simplices of $\mathscr{C}$ is precisely the datum that we need in order to define the homotopy coherencies of the ∞-functor represented by F.

(2) For *any* simplicial set K and for a fixed quasi-category $\mathscr{C}$ the simplicial mapping complex

$$\underline{\mathrm{Map}}^\bullet_{\mathrm{sSet}}(K, \mathscr{C}) \colon [n] \mapsto \mathrm{Hom}_{\mathrm{sSet}}(K \times \Delta^n, \mathscr{C})$$

is a quasi-category ([Lur09, Proposition 1.2.7.3]). As a consequence, for any simplicial set (and, in particular, for any quasi-category) K, we shall define

$$\mathrm{Fun}(K, \mathscr{C}) := \mathrm{Map}^\bullet_{\mathrm{sSet}}(K, \mathscr{C})$$

to be the *quasi-category of quasi-functors between* K *and* $\mathscr{C}$. It is a striking result that, differently from what happens in the theory of ordinary model categories, this quasi-category models the homotopy theory of functors between the underlying homotopy categories. In other words, if $\mathscr{C}$ and $\mathscr{D}$ are quasi-categories, we have an equivalence of homotopy categories

$$\mathrm{Fun}(\mathrm{h}\mathscr{C}, \mathrm{h}\mathscr{D}) \simeq \mathrm{hFun}(\mathscr{C}, \mathscr{D}),$$

where the homotopy category $\mathrm{h}\mathscr{C}$ of a quasi-category is defined as in Construction 2.1.11.

(3) In ∞-category theory, one can easily generalize the notions of *essentially surjective* and *fully faithful* ∞-functors: they are functors $\mathscr{C} \to \mathscr{D}$ which are surjective on the homotopy equivalence classes of objects, and functors $\mathscr{C} \to \mathscr{D}$ such that

$$\mathrm{Map}_{\mathscr{C}}(x, y) \simeq \mathrm{Map}_{\mathscr{D}}(F(x), F(y))$$

for any couple of objects x and y of $\mathscr{C}$, respectively. It makes *less* sense to define "full" or "faithful" ∞-functors: the fact that we have a *homotopy type* of maps between objects makes it almost meaningless to talk about "injectivity" and "surjectivity" on the homotopy type of maps.

Construction 2.1.11. As in the case of simplicially enriched and topologically enriched categories, there is also a notion of a *homotopy* 1*-category underlying a quasi-category* $\mathscr{C}$, which agrees with the homotopy category of its underlying simplicial set defined through the functor 1.4.4. In the case $\mathscr{C}$ is a quasi-category, the homotopy category $\mathrm{h}\mathscr{C}$ can be equivalently characterized as follows.

(1) Any vertex of $\mathscr{C}$ yields an object of $\mathrm{h}\mathscr{C}$.
(2) Any edge $f : \Delta^1 \to \mathscr{C}$ yields an arrow between the source $\mathrm{d}^1_0(f) =: x$ and the target $\mathrm{d}^1_1(f) =: y$. The identity on an object x is the degenerate 1-simplex $\mathrm{s}^1_0(x) =: \mathrm{id}_x$.
(3) We mod out the set of edges of $\mathscr{C}$ by the homotopy equivalence relations, that identify all edges $f : x \to y$ and $g : x \to y$ whenever there exists a 2-simplex making the following diagram commute.

$$\begin{tikzcd}
 & y \arrow[dr, equal, "\mathrm{id}_y"] \arrow[d, phantom, "\Downarrow"] & \\
x \arrow[ur, "f"] \arrow[rr, "g"'] & {} & y.
\end{tikzcd}$$

Notice that with such a definition of the homotopy category of a quasi-category $\mathscr{C}$, the definition of equivalence provided in Remark 2.1.8.(5) can be restated as follows: $f : x \to y$ is an equivalence precisely if it represents an isomorphism in the homotopy category.

This discussion allows us to define what an equivalence of quasi-categories is.

Definition 2.1.12. An *equivalence of quasi-categories* is a quasi-functor $F : \mathscr{C} \to \mathscr{D}$ such that the induced functor $\mathrm{h}F : \mathrm{h}\mathscr{C} \to \mathrm{h}\mathscr{D}$ is essentially surjective, and the induced map of mapping complexes

$$\mathrm{Map}_{\mathscr{C}}(x, y) \longrightarrow \mathrm{Map}_{\mathscr{D}}(F(x), F(y))$$

is a weak equivalence of Kan complexes.

Remark 2.1.13. The meaning of Definition 2.1.12 is that an equivalence of quasi-categories is an equivalence on the underlying homotopy categories with sufficient homotopy coherence data to lift it to a proper functor of quasi-categories. At first glance, people who are not accustomed to homotopy theory may find this definition too weak; yet this is precisely the right homotopy generalization of what an equivalence between categories should be. Namely, let $\mathbb{I}$ be the *walking isomorphism category*, i.e., the (quasi-category associated to the) category consisting of two objects $[0]$ and $[1]$, and two non-trivial arrows $f : [0] \to [1]$ and $f^{-1} : [1] \to [0]$ subject to the relations $f \circ f^{-1} = \mathrm{id}_{[1]}$ and $f^{-1} \circ f = \mathrm{id}_{[0]}$. Then a quasi-functor of quasi-categories $F : \mathscr{C} \to \mathscr{D}$ is an equivalence in the sense of Definition 2.1.12 if and only if one of the two following equivalent conditions hold.

(1) For any quasi-category $\mathscr{E}$ the composition functor $F_*\colon \operatorname{Fun}(\mathscr{E}, \mathscr{C}) \to \operatorname{Fun}(\mathscr{E}, \mathscr{D})$ is an equivalence of quasi-categories.
(2) The functor F can be extended to a *homotopy coherent equivalence*, i.e., there exists a quasi-functor $G\colon \mathscr{D} \to \mathscr{C}$ and commuting diagrams

$$\begin{tikzcd} & & \mathscr{C} \\ \mathscr{C} \arrow[urr, bend left, equal, "\mathrm{id}_{\mathscr{C}}"] \arrow[rr, "\alpha"] \arrow[drr, bend right, "G \circ F"'] & & \operatorname{Fun}(\mathbb{I}, \mathscr{C}) \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ & & \mathscr{C} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & & \mathscr{D} \\ \mathscr{D} \arrow[urr, bend left, equal, "\mathrm{id}_{\mathscr{D}}"] \arrow[rr, "\beta"] \arrow[drr, bend right, "F \circ G"'] & & \operatorname{Fun}(\mathbb{I}, \mathscr{D}) \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ & & \mathscr{D}. \end{tikzcd}$$

See also [RV22, Theorem 1.4.7].

The homotopy category of a quasi-category has all the expected properties of the homotopy category of a category enriched in homotopy types.

Proposition 2.1.14 ([Lur09, Propositions 1.2.3.1 and 1.2.5.1]). *The functor* $\mathrm{h}\colon \mathrm{sSet} \to \mathrm{Cat}$ *is a left adjoint, which is right adjoint to the nerve functor* $\mathrm{N}\colon \mathrm{Cat} \to \mathrm{sSet}$*. Moreover, a quasi-category* $\mathscr{C}$ *is a Kan complex if and only if* $\mathrm{h}\mathscr{C}$ *is a groupoid.*

We conclude this section with a brief remark on the history and the heuristics of quasi-categories. Quasi-categories were first introduced by Boardman and Vogt, under the term *weak Kan complexes*, in their study of homotopy invariant algebraic structures, and were later studied in greater detail by Joyal starting from the 1980s. According to Joyal himself, he wished to extend category theory to weak Kan complexes – this is why he called them *quasi-categories* – but he stopped his project for almost fifteen years because of difficulties encountered while trying to prove that a quasi-category is always a Kan complex if its homotopy category is a groupoid. He finally achieved this result around the second half of the 1990s; later, he exhibited a model structure on the category of simplicial sets whose fibrant-cofibrant objects were precisely quasi-categories (see Theorem 2.3.6). Some years later, Lurie developed the theory of quasi-categories in the seminal [Lur09], using Joyal's theory (and unpublished ideas by Rezk) as a stepping stone for his research. Nowadays, topologically enriched categories, simplicially enriched categories and quasi-categories, (but also complicial setes, complete Segal spaces, and so forth) are only meant to be interpreted as various *models* for ∞-category theory, all of which enjoy the same properties up to formalism issues. The choice of the quasi-categorical model is motivated purely by the technical advantages they bring to the table when proving *foundational results and theorems*, and nothing else: the homotopy theories of topologically enriched categories, simplicially enriched categories, and quasi-categories (but also complete Segal spaces, complicial sets...) are all equivalent one to the other. For instance, there are both *topological*

$$\mathrm{N}_{\mathrm{top}}\colon \mathrm{Cat}_{\mathrm{Top}} \longrightarrow \mathrm{sSet}_{\mathrm{Joyal}} \tag{2.1.15}$$

and *simplicial*

$$\mathrm{N}_\Delta : \mathrm{Cat}_\Delta \longrightarrow \mathrm{sSet}_{\mathrm{Joyal}} \tag{2.1.16}$$

nerve functors, that produce a quasi-category out from a topologically or simplicially enriched category, respectively. Viceversa, one can extract a simplicially enriched category from any quasi-category (actually, from any simplicial set) via a functor

$$\mathfrak{C} : \mathrm{sSet}_{\mathrm{Joyal}} \longrightarrow \mathrm{Cat}_\Delta. \tag{2.1.17}$$

Hence, by composing with Sing, we can obtain a topologically enriched category $\mathrm{Sing}(\mathfrak{C}[\mathscr{C}])$. This assignation produces an adjunction which yields an equivalence at the level of homotopy categories between Cat_Δ (hence $\mathrm{Cat}_{\mathrm{Top}}$) and sSet ([Lur09, Theorem 1.1.5.13]). Moreover, for any quasi-category $\mathscr{C}$, the functor $\mathfrak{C}$ induces a natural equivalence at the level of homotopy categories, i.e.,

$$\mathrm{h}\mathscr{C} \simeq \pi_0(\mathfrak{C}(\mathscr{C})).$$

Here π_0 is the homotopy category functor for simplicially enriched categories 2.1.6. Long story short: even if we shall stick to the model provided by quasi-categories (if we need one), we can transfer all our constructions from one setting to another essentially using the functors 2.1.15, 2.1.16 and 2.1.17. For all these reasons, the reader is *deeply encouraged* to think of this theory in model-independent terms.

2.2. **Category theory for ∞-categories: an introduction.** There are many important concepts that we would like to preserve, or suitably generalize, from ordinary category theory to the setting of ∞-category theory. Among the most important, we recall the following.

(1) We want to arrange ∞-categories in some $(\infty, 2)$-*category of ∞-categories*, up to size issues.
(2) We want to deal with *adjunctions*.
(3) Of course, we want to be able to talk about *limits* and *colimits* of arbitrary shape in ∞-categories.
(4) We want to prove an analogue of *Ninja Yoneda Lemma* (Theorem 1.4.1).

In the writer's opinion, nowadays it is a bit anachronistic to base the statements and the proofs of the theory *entirely* on the formalism of quasi-categories developed by Joyal and Lurie. The reason is the following: recently, there has been a surge of results dealing with the theory of enriched ∞-category theory and of (∞, n)-categories that allow a way more *intrinsic* treatment of the theory of ∞-categories, for example via the theory of ∞-cosmoi developed in [RV22], which moreover highlights the naturality of the definitions in the higher categorical world and almost makes rigorous the following slogan.

Slogan 2.2.1. Basic constructions of ordinary 1-category theory can be carried out almost *verbatim* in the ∞-categorical world, up to replacing *commutative diagrams* with *homotopy coherent diagrams*, *isomorphisms* with *homotopy equivalences*, *unique up to a unique isomorphism* with *unique up to a contractible space of homotopies*, and *sets* with *spaces* or *homotopy types*.

If a reader finds this slogan believable enough and is only interested in using ∞-categorical tools for algebra and geometry, they can safely skip these sections and jump directly to Section 3. Otherwise, let us begin!

2.3. **The $(\infty, 2)$-category of ∞-categories.** For the moment, we shall stick to the quasi-categorical model in order to explicitly present two fundamental ∞-categories: the ∞-category of spaces $\mathcal{S}$, and the ∞-category of ∞-categories Cat_∞. The former will play the role of the category Set of sets: all locally small ∞-categories are enriched over $\mathcal{S}$, they are the most prototypical example of a ∞-topos, and every cocomplete ∞-category is naturally tensored over $\mathcal{S}$; the latter is, simply, the ambient world for our ∞-categories. First of all, we need a small detour: we have to show how we can produce a quasi-category from a model category, or in general, from a category endowed with a class $\mathscr{W}$ of weak equivalences.

Definition 2.3.1 (Hammock or Dwyer-Kan localization, [DK80, Definition 2.1])**.** Let $(\mathscr{C}, \mathscr{W})$ be the datum of a category $\mathscr{C}$ with a class of weak equivalences $\mathscr{W}$ (i.e., a class of morphisms that contain all isomorphisms and satisfy the two-out-of-three property). The *hammock* or *Dwyer-Kan* localization of $\mathscr{C}$ at $\mathscr{W}$ is the simplicially enriched category $\mathrm{L}^{\mathrm{H}}(\mathscr{C}, \mathscr{W})$ described as follows.

(1) Objects of $\mathrm{L}^{\mathrm{H}}(\mathscr{C}, \mathscr{W})$ are the same as the objects of $\mathscr{C}$.
(2) The simplicial set of maps from X to Y in $\mathrm{L}^{\mathrm{H}}(\mathscr{C}, \mathscr{W})$ is described in degree k by *hammocks of width k and any length*. Namely: the k-simplices of $\mathrm{Hom}^{\bullet}_{\mathrm{L}^{\mathrm{H}}(\mathscr{C}, \mathscr{W})}(X, Y)$ are given by diagrams of the form

$$\begin{tikzcd}
 & C_{0,1} \arrow[r] \arrow[ddl] \arrow[d] & C_{0,2} \arrow[d] & \cdots \arrow[l] & C_{0,n} \arrow[l] \arrow[d] \arrow[ddr] & \\
 & C_{1,1} \arrow[r] \arrow[dl] \arrow[d] & C_{1,2} \arrow[d] & \cdots \arrow[l] & C_{1,n} \arrow[l] \arrow[d] \arrow[dr] & \\
X & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & Y \\
 & C_{k,1} \arrow[r] \arrow[ul] & C_{k,2} & \cdots \arrow[l] & C_{k,n} \arrow[l] \arrow[ur] &
\end{tikzcd}$$

subject to the following relations.
- n ranges over all non-negative integers;
- all vertical arrows and all horizontal arrows pointing to the left lie in $\mathscr{W}$;
- any two consecutive horizontal arrows point toward different directions;
- no column consists *only* of identity maps.

Faces and degeneracies are then defined in the natural way.

Remark 2.3.2. If $\mathscr{C}$ is moreover a *model* category, it is sufficient to consider only fibrant and cofibrant objects.

It is quite clear from Definition 2.3.1 that the class of vertices of the simplicial set of maps between X and Y in $\mathrm{L}^{\mathrm{H}}(\mathscr{C}, \mathscr{W})$ is pretty close to the class of arrows from X to Y in the localization $\mathscr{C}\left[\mathscr{W}^{-1}\right]$ of Definition 1.2.1. And indeed:

Proposition 2.3.3 ([DK80, Proposition 3.1]). *There exists an equivalence of categories*
$$\pi_0\big(\mathrm{L}^{\mathrm{H}}(\mathscr{C},\mathscr{W})\big)\simeq\mathscr{C}\big[\mathscr{W}^{-1}\big],$$
where π_0 is the homotopy category functor for simplicially enriched categories 2.1.6.

Taking the simplicial nerve $\mathrm{N}_\Delta\big(\mathrm{L}^{\mathrm{H}}(\mathscr{C},\mathscr{W})\big)$ (2.1.16), we obtain a honest quasi-category. The heuristics of what we are doing is clear: we are enhancing our model category to a ∞-category by considering simultaneously *all possible homotopy coherence data* that remember *how* diagrams commute in the homotopy category $\mathscr{C}\big[\mathscr{W}^{-1}\big]$. Moreover, the construction of Definition 2.3.1 provides an immediate candidate for the ∞-category of spaces $\mathcal{S}$: it is the ∞-category associated to the standard model structure on Top, or equivalently[1] on sSet.

Definition 2.3.4. The ∞-*category of spaces* (or *of homotopy types*) is the ∞-category presented by the quasi-category
$$\mathcal{S}:=\mathrm{N}_\Delta\big(\mathrm{L}^{\mathrm{H}}(\mathrm{Top},\mathscr{W})\big).$$

Remark 2.3.5. Definition 2.3.4 makes the *homotopy hypothesis* trivially true: since ($(n+1)$-coskeletal) Kan complexes correspond to the nerve of (n-)groupoids, this definition identifies the homotopy theory of n-homotopy types with the higher categorical theory of n-groupoids, essentially by standard model theoretic arguments. However, this identification can become less obvious, and even highly non trivial, with other models: see [nLa23b] for more details.

In a similar fashion to Definition 2.3.4, we can define the ∞-category of ∞-categories to be the Dwyer-Kan localization of some model category.

Theorem 2.3.6 ([Joy02; Lur09]). *There is a model structure on the category of simplicial sets where cofibrations are monomorphisms, and weak equivalences are detected via the functor $\mathfrak{C}$ (2.1.17). The fibrant and cofibrant objects of this model category are precisely the quasi-categories, i.e.,* $\mathrm{sSet}_{\mathrm{Joyal}}$. *Moreover, such model category is combinatorial*[2] *and it is monoidal with respect to the standard Cartesian monoidal structure on* sSet[3].

Remark 2.3.7. Theorem 2.3.6 allows us to consider, modulo suitably enlarging our universe, a *quasi-category of quasi-categories*, which is the quasi-category associated to $\mathrm{sSet}_{\mathrm{Joyal}}$. Again, let us remark that such quasi-category has to be interpreted as a mere *model* for a ∞-category of ∞-categories, that we shall denote either by Cat_∞ (for the large ∞-category of small ∞-categories) or $\widehat{\mathrm{Cat}}_\infty$ (for the huge ∞-category of large ∞-categories). Moreover, the compatibility of the model and the monoidal structures allows us to consider the ∞-categories Cat_∞ and $\widehat{\mathrm{Cat}}_\infty$ to be enriched over themselves. This is another way to interpret the fact that they are in fact $(\infty,2)$-*categories*: the ∞-category of morphisms between two ∞-categories $\mathscr{C}$ and $\mathscr{D}$ is simply its internal mapping object $\mathrm{Fun}(\mathscr{C},\mathscr{D})$. This mirrors the definition of a $(2,2)$-category as an ordinary category which is (weakly) enriched over the Cartesian monoidal category Cat.

[1] Until now, this *equivalently* is a bit of a stretch, but we will justify the use of this word in Theorem 2.4.3.

[2] This simply means that it is tractable and generated by small enough data, in some sense.

[3] This is simply a technical conditions that guarantees that both the product of simplicial sets *and* the internal mapping complex pass at the level of the homotopy category.

So, we have gathered our ∞-categories in a $(\infty, 2)$-category $\widehat{\mathrm{Cat}}_\infty$. Nice! We can finally start doing *real* category theory, and there is more: the technical machinery of ordinary 2-category theory which has been recently developed in the ∞-categorical setting allows us to define, in a completely abstract and theoretical fashion, concepts such as adjunctions, limits and colimits, highlighting their similarity with the usual definitions encountered in ordinary category theory. From now one, we shall drop the *quasi* before *quasi-categories* and replace it with ∞.

2.4. **Adjunctions.** In [Lur09, Definition 5.2.2.1], Lurie argues that an adjunction between ∞-categories $\mathscr{C}$ and $\mathscr{D}$ should be the datum of an ∞-functor $\mathscr{M} \to \Delta^1$, which is both a Cartesian and coCartesian fibration, together with equivalences

$$\mathscr{M} \times_{\Delta^1} \{[0]\} \simeq \mathscr{C}$$

and

$$\mathscr{M} \times_{\Delta^1} \{[1]\} \simeq \mathscr{D}.$$

While we shall discuss Cartesian and coCartesian fibrations in Section 2.7, we have still not defined them yet; moreover, in ordinary category theory adjunctions are not defined in this way. Hence, we prefer to follow the approach of [RV22], which recovers the same theory of adjunctions between ∞-categories as Lurie's, but it is way more recognizable from a 1-categorial point of view.

Definition 2.4.1 ([RV22, Definition 2.1.1])**.** An *adjunction between ∞-categories* is the datum of two ∞-functors

$$F\colon \mathscr{C} \rightleftarrows \mathscr{D} \colon G$$

together with a pair of natural transformations $\eta\colon \mathrm{id}_{\mathscr{D}} \Rightarrow G \circ F$ and $\epsilon\colon F \circ G \Rightarrow \mathrm{id}_{\mathscr{C}}$ satisfying the following triangular identities in the underlying homotopy 2-category $\mathrm{h}\,\widehat{\mathrm{Cat}}_\infty$.

$$\begin{tikzcd} & \mathscr{C} \arrow[rr, equal, "\mathrm{id}_{\mathscr{C}}"] \arrow[dr, "F"] \arrow[d, phantom, "\epsilon\Downarrow"] & & \mathscr{C} \\ \mathscr{D} \arrow[ur, "G"] \arrow[rr, equal, "\mathrm{id}_{\mathscr{D}}"'] & {} & \mathscr{D} \arrow[ur, "G"'] \arrow[u, phantom, "\Downarrow\eta"] & \end{tikzcd} \;\overset{\simeq}{=}\; \begin{tikzcd} \mathscr{C} \\ \mathscr{D} \arrow[u, bend left, "G"] \arrow[u, bend right, "G"'] \arrow[u, phantom, "="] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathscr{C} \arrow[rr, equal, "\mathrm{id}_{\mathscr{C}}"] \arrow[dr, "F"'] & {} \arrow[d, phantom, "\eta\Downarrow"] & \mathscr{C} \arrow[dr, "F"] & \\ & \mathscr{D} \arrow[ur, "G"] \arrow[rr, equal, "\mathrm{id}_{\mathscr{D}}"'] & {} \arrow[u, phantom, "\Downarrow\epsilon"] & \mathscr{D} \end{tikzcd} \;\overset{\simeq}{=}\; \begin{tikzcd} \mathscr{C} \arrow[d, bend right, "F"'] \arrow[d, bend left, "F"] \arrow[d, phantom, "="] \\ \mathscr{D} \end{tikzcd} .$$

In this situation, we shall say that F is left adjoint to G and G is right adjoint to F and we shall denote it by $F \dashv G$ (or equivalently $G \vdash F$).

Let us remark that, in spite of a vast portion of literature still adopting the definition *à la* Lurie, this is the definition that Joyal himself had proposed for an adjunction between quasi-categories. It is a remarkable fact that, in order to provide an adjunction at the level of ∞-categories, one needs to specify only 2-dimensional data: the rest is recovered – and in a homotopically essentially unique way! – from these, see [RV16, Theorem 4.4.18].

Proposition 2.4.2 ([RV22, Sections 2.1 and 4.1])**.**

(1) If $F\colon \mathscr{C} \rightleftarrows \mathscr{D}\colon G$ is an adjunction, then for any ∞-category $\mathscr{E}$ there is an induced adjunction

$$F_*\colon \mathrm{Fun}(\mathscr{E}, \mathscr{C}) \rightleftarrows \mathrm{Fun}(\mathscr{E}, \mathscr{D})\colon G_*.$$

(2) *If $F\colon \mathscr{C} \rightleftarrows \mathscr{D}\colon G$ and $F'\colon \mathscr{D} \rightleftarrows \mathscr{E}\colon G'$ are adjunctions, then $F' \circ F\colon \mathscr{C} \rightleftarrows \mathscr{E}\colon G \circ G'$ is an adjunction as well.*
(3) *If an ∞-functor $F\colon \mathscr{C} \to \mathscr{D}$ admits two right adjoints G and G', then $G \simeq G'$. An analogous claim holds for F admitting two left adjoints.*
(4) *Any equivalence of ∞-categories can be modified into an adjoint equivalence up to modifying one of the natural equivalences α and β of Remark 2.1.13.(2).*
(5) *A pair of ∞-functors $F\colon \mathscr{C} \rightleftarrows \mathscr{D}\colon G$ is an adjunction of ∞-categories if and only if for any object C in $\mathscr{C}$ and any object D in $\mathscr{D}$ it induces a natural equivalence of mapping spaces*

$$\mathrm{Map}_{\mathscr{D}}(F(C), D) \xrightarrow{\simeq} \mathrm{Map}_{\mathscr{C}}(C, G(D)).$$

Moreover, this equivalence is induced by applying $\eta_C \circ G$.

Proposition 2.1.14 guarantees that adjunctions in the ∞-categorical world behave in a very similar way to the ones in the 1-categorical framework. Actually, there is *more*: if two ∞-categories arise from the Dwyer-Kan localization of two model categories (Definition 2.3.1), then Quillen adjunctions and equivalences are turned into ∞-categorical adjunctions and equivalences.

Theorem 2.4.3 ([Maz16, Theorem 2.1])**.** *Given a Quillen adjunction $F\colon \mathscr{C} \rightleftarrows \mathscr{D}\colon G$ between model categories, then the restrictions*

$$F|_{\mathscr{C}_{\mathrm{c}}}\colon \mathscr{C}_{\mathrm{c}} \longrightarrow \mathscr{D}$$

and

$$F|_{\mathscr{D}_{\mathrm{f}}}\colon \mathscr{D}_{\mathrm{f}} \longrightarrow \mathscr{C}$$

induce an adjunction between their associated ∞-categories. If moreover $F \dashv G$ is a Quillen equivalence, then the induced ∞-functors yield an equivalence of ∞-categories.

2.5. **Limits and colimits.** In most handbooks of categorical algebra limits and colimits are presented *before* adjunctions. This is the approach of [Lur09] as well: we briefly review it here.

(1) First, Lurie generalizes the *join operation* of sets to quasi-categories in [Lur09, Section 1.2.8]. In ordinary category theory, the join of two categories $\mathscr{C} \star \mathscr{D}$ is the category whose objects are given by the disjoint union of the classes of objects of $\mathscr{C}$ and $\mathscr{D}$, and whose class of arrows is described as follows. If X and Y are objects in $\mathscr{C}$, then the set of maps between X and Y is just the set of maps between X and Y in $\mathscr{C}$; an analogous claim holds if X and Y are both in $\mathscr{D}$. If X belongs to $\mathscr{C}$ and Y belongs to $\mathscr{D}$ then there exists a unique map $X \to Y$ in $\mathscr{C} \star \mathscr{D}$, while if X belongs to $\mathscr{D}$ and Y belongs to $\mathscr{C}$ there is no map at all.
(2) In this way, for any diagram $p\colon K \to \mathscr{C}$ in a quasi-category $\mathscr{C}$ he is able to define *over-quasi-categories* $\mathscr{C}_{/p}$ and *under-quasi-categories* $\mathscr{C}_{p/}$ as those quasi-categories such that, for any other simplicial set Y, there is an identity of mapping complexes

$$\mathrm{Map}_{\mathrm{sSet}}\big(Y, \mathscr{C}_{/p}\big) = \mathrm{Map}_p(Y \star K, \mathscr{C}),$$

where the subscript on the right hand side denotes that we are considering only maps of simplicial sets whose restriction to K agrees with p. This is the content of [Lur09, Section 1.2.9]

(3) Next, Lurie introduces *final objects* in a quasi-category $\mathscr{C}$ ([Lur09, Section 1.2.12]): an object $\mathbb{1}$ in a quasi-category $\mathscr{C}$ is final if and only if for any other object X of $\mathscr{C}$ the mapping space $\mathrm{Map}_{\mathscr{C}}(X, \mathbb{1})$ is contractible. Dually, an *initial object* is just a final object in the opposite quasi-category $\mathscr{C}^{\mathrm{op}}$.

(4) Finally, a limit for a diagram $p\colon K \to \mathscr{C}$ is defined to be a final object in $\mathscr{C}_{/p}$, and dually a colimit is defined to be an initial object in $\mathscr{C}_{p/}$ ([Lur09, Definition 1.2.13.4]).

However, having already developed the theory of adjunctions in the ∞-categorical setting, we can simplify *a lot* our definition of limits and colimits.

Definition 2.5.1. Let K be any simplicial set. We say that an ∞-category $\mathscr{C}$ admits *colimits of shape K* if the ∞-functor

$$\mathrm{const}\colon \mathscr{C} \simeq \mathrm{Fun}(\{*\}, \mathscr{C}) \longrightarrow \mathrm{Fun}(K, \mathscr{C}),$$

induced by restriction along the canonical map $K \to \{*\}$, admits a left adjoint

$$\mathrm{colim}\colon \mathrm{Fun}(K, \mathscr{C}) \longrightarrow \mathscr{C}.$$

Dually, we say that an ∞-category $\mathscr{C}$ *admits limits of shape K* if the above ∞-functor admits a right adjoint

$$\lim\colon \mathrm{Fun}(K, \mathscr{C}) \longrightarrow \mathscr{C}.$$

Given a diagram $p\colon K \to \mathscr{C}$, seen as an object in $\mathrm{Fun}(K, \mathscr{C})$, its colimit (resp. its limit) is the image of p under the left adjoint colim (resp., under the right adjoint lim).

Notice that Definition 2.5.1 already extracts *a lot* of information on the behavior of limits and colimits in ∞-categories. You can toy around with Proposition 2.4.2 to extract useful and desired properties, such as the fact that the space of limits of a homotopy coherent diagram is either empty or contractible, or the fact that limits are final in the ∞-category of cones over a homotopy coherent diagram, or again that limits are preserved by left adjoints, as well as their dual statements for the case of colimits. Of course, this comes with a drawback: we are not allowing our ∞-categories to have some, *but not all*, limits or colimits of a certain shape. We provide a 2-categorical definition just for reference, but we guarantee that in the following we shall need only ∞-categories admitting *all* limits or colimits of a certain shape – most often than not, even admitting *all* sufficiently small limits or colimits.

Definition 2.5.2 ([RV22, Definition 2.3.8])**.** Let K be a simplicial set, and let $d\colon D \to \mathrm{Fun}(K, \mathscr{C})$ be a family of diagrams of shape K in $\mathscr{C}$. We say that the family D *admits a colimit in $\mathscr{C}$* if there exists an ∞-functor $\mathrm{colim}\colon D \to \mathscr{C}$ and a 2-simplex

$$\begin{tikzcd} & \mathscr{C} \arrow[d, "\mathrm{const}"] \\ D \arrow[ur, bend left, "\mathrm{colim}"] \arrow[r, "d"'] \arrow[ur, phantom, "\Uparrow \eta"] & \mathrm{Fun}(K, \mathscr{C}) \end{tikzcd}$$

such that any 2-simplex of the following form

$$\begin{tikzcd} \mathscr{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Uparrow\sigma"] & \mathscr{C} \arrow[d, "\mathrm{const}"] \\ D \arrow[r, "d"'] & \mathrm{Fun}(K, \mathscr{C}) \end{tikzcd}$$

factors uniquely through η. Dually, we say that the family D *admits a limit in* $\mathscr{C}$ if there exists an ∞-functor $\lim\colon D \to \mathscr{C}$ and a 2-simplex

$$\begin{tikzcd} & \mathscr{C} \arrow[d, "\mathrm{const}"] \\ D \arrow[ur, bend left, "\lim"] \arrow[r, "d"'] \arrow[ur, phantom, "\Downarrow\eta"] & \mathrm{Fun}(K, \mathscr{C}) \end{tikzcd}$$

such that any 2-simplex of the following form

$$\begin{tikzcd} \mathscr{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow\sigma"] & \mathscr{C} \arrow[d, "\mathrm{const}"] \\ D \arrow[r, "d"'] & \mathrm{Fun}(K, \mathscr{C}) \end{tikzcd}$$

factors uniquely through ϵ.

Remark 2.5.3. Notice that if $\mathscr{C}$ is a model category, its associated ∞-category $\mathrm{N}_\Delta\big(\mathrm{L}^{\mathrm{H}}(\mathscr{C}, \mathscr{W})\big)$ is both complete and cocomplete, and limits and colimits in $\mathrm{N}_\Delta\big(\mathrm{L}^{\mathrm{H}}(\mathscr{C}, \mathscr{W})\big)$ are modeled by homotopy limits and homotopy colimits in $\mathscr{C}$. Try to prove this statement yourself by concatenating Definition 2.5.1 and Example 1.2.7 with Theorem 2.4.3.

Corollary 2.5.4. *The ∞-categories $\mathcal{S}$, Cat_∞ and $\widehat{\mathrm{Cat}}_\infty$ are complete and cocomplete, and the inclusions $\mathcal{S} \subseteq \mathrm{Cat}_\infty$ and $\widehat{\mathcal{S}} \subseteq \widehat{\mathrm{Cat}}_\infty$ preserve all limits and colimits.*

Remark 2.5.5 ([Lur09, Remark 1.2.5.6])**.** Actually, Theorem 2.4.3 implies something more. The identity functor on simplicial sets

$$\mathrm{id}_{\mathrm{sSet}}\colon \mathrm{sSet} \longrightarrow \mathrm{sSet},$$

where the source is endowed with the Quillen standard model structure of Theorem 1.3.9 and the target with the Joyal model structure of Theorem 2.3.6, is both left and right Quillen. Indeed, it preserves fibrant-cofibrant objects (since Kan complexes are in particular quasi-categories, while all simplicial sets are cofibrant), weak equivalences (since $\mathfrak{C}$ sends a weak equivalence of simplicial sets to a weak equivalence of simplicial categories) and all limits and colimits (trivially). In particular, the inclusion ∞-functor $\mathcal{S} \subseteq \mathrm{Cat}_\infty$ is *both a left and a right adjoint*. Its right adjoint is the *core ∞-functor*

$$(-)^{\simeq}\colon \mathrm{Cat}_\infty \longrightarrow \mathcal{S}$$

which is informally described by sending an ∞-category $\mathscr{C}$ to its core, i.e., its maximal sub-∞-groupoid $\mathscr{C}^{\simeq}$. On the other hand, the left adjoint

$$\mathrm{gpdfy}\colon \mathrm{Cat}_{\infty} \longrightarrow \mathcal{S}$$

is the *groupoidification ∞-functor*, which completes an ∞-category to an ∞-groupoid by adding inverses to all non-invertible morphisms. The groupoidification ∞-functor is modeled by the fibrant replacement of a quasi-category for the standard Quillen model structure on simplicial sets, while the core ∞-functor is modeled by a *left Bousfield localization*, which is the classical model-theoretic way to present the homotopy datum of a reflective full subcategory of a homotopy category (a reference for the standard model-theoretic definition is [Hir03, Definition 3.3.1]). These arguments can be carried out *verbatim* also for their "huge" counterpart.

Example 2.5.6 (Notable examples and properties of limits and colimits).

(1) We can consider arbitrary *products* and *coproducts* in an ∞-category $\mathscr{C}$: in particular, the *terminal object* of $\mathscr{C}$ is the product on the empty ∞-category and the *initial object* is the coproduct on the empty∞-category. We say that $\mathscr{C}$ is *pointed* if both initial and final objects exist, and they are naturally equivalent. Note that in the case of the ∞-category of spaces, since $\mathcal{S}$ can be modeled by the model category Top where every object is fibrant, products in $\mathcal{S}$ are precisely modeled by products of topological spaces. (This is not true in sSet, since fibrant objects are precisely Kan complexes; anyway, one does not need to fibrantly replace the factors in a product when they are finite, since finite products of simplicial sets preserve weak equivalences.)

(2) We can consider *pullbacks* and *pushouts*. Here, if $\mathscr{C}$ is a Dwyer-Kan localization of a model category, the things become trickier: even when a span or cospan of objects in $\mathscr{C}$ is comprised of both fibrant and cofibrant objects, the arrows can fail to be fibrations or cofibrations, and so one needs to replace the diagram with one featuring at least a fibration (in the case of pullbacks) or a cofibration (in the case of pushout). In particular, pullbacks and pushouts of ∞-categories provide a generalization of *homotopy* pullbacks and pushouts.

(3) We can consider *filtered colimits*. While the formal definition of a filtered diagram is a bit more convoluted than in the classical case (even if the idea is precisely the same: a diagram K is filtered if every small sub-diagram $J \subseteq K$ admits a cocone in K), for us it will be sufficient to characterize filtered colimits as *precisely* those colimits which commute with any finite limit in $\mathscr{C}$ ([Lur09, Proposition 5.3.3.3]). Notice that in almost all model categories we encountered until now, cofibrations are (some subclass of) monomorphisms and inclusions, so filtered colimits in $\mathcal{S}$ and in Cat_{∞} (but also in more algebraic ∞-categories that we will study in Section 3) model some sort of filtered object in the "concrete" sense. The dual notion of a filterd colimit is provided by *inverse limits*, i.e., limits of cofiltered diagrams.

(4) We can consider *geometric realizations*, i.e., colimits of diagrams of the form $\Delta^{\mathrm{op}} \to \mathscr{C}$. In ordinary 1-category, one of the most important kind of colimits is represented by the class of *reflexive coequalizers*, which extend the theory of quotients by equivalence relations. In ∞-category, reflexive coequalizers are not enough anymore to capture all

the homotopy coherencies that homotopy quotients must satisfy, because they are "too low dimensional". Simplicial diagrams provide the correct generalization of coequalizers in ∞-category: indeed, truncating a diagram in degrees $\leqslant 1$ (i.e., by pre-composing with the natural inclusion $\Delta^{\mathrm{op}}_{\leqslant 1} \subseteq \Delta^{\mathrm{op}}$) one obtains precisely a coequalizer diagram. For instance, when we will be able to talk about *group objects* in a ∞-category, we shall see that the quotient of the action of a group G on an object X in $\mathscr{C}$ is encoded by the geometric realization

$$[X/G] := \operatorname*{colim}_{n \in \mathbb{N}} \Big(\dots \; G^{\times 3} \times X \; G^{\times 2} \times X \; G \times X \; X \Big),$$

where the right-oriented maps are given by all possible binary products in G, all possible binary actions of G on X, and projections. The dual notion of geometric realizations is provided by *totalizations*, i.e., limits of cosimplicial diagrams $\Delta \to \mathscr{C}$.

(5) In general one can talk about *sifted colimits*, i.e., colimits of diagrams of shape K where K is a sifted ∞-category. We will not actually need the definition of what *sifted* means, but we are interested in the fact that Δ^{op} is sifted ([Lur09, Proposition 5.5.8.4]), and that sifted colimits are precisely those colimits which commute with finite products ([Lur09, Proposition 5.5.8.11]); in particular, filtered colimits are sifted as well. Actually *sifted colimits are essentially of these two types*: i.e., to have all sifted colimits it is sufficient to have geometric realizations and filtered colimits (this is stated, somewhat cryptically, in [Lur09, Lemma 5.5.8.14] and spelled out in a bit more detail in [Lur09, Corollary 5.5.8.17]).

(6) We shall say that $\mathscr{C}$ is *(co)complete* if it admits all (co)limits, and we shall say that it is *finitely (co)complete* if it admits all finite (co)limits, just as in the classical case. As in ordinary category theory, an ∞-category $\mathscr{C}$ is finitely cocomplete if and only if it admits pushouts and an initial object ([Lur09, Corollary 4.4.2.4]), or equivalently finite coproducts and geometric realizations ([Lur17, Lemma 1.3.3.10]). Moreover, $\mathscr{C}$ is cocomplete if and only if it admits all finite coproducts and sifted colimits (combine [Lur17, Lemma 1.3.3.10] with [Lur09, Corollary 4.2.3.11]), or if it admits all coproducts and pushouts ([Lur09, Proposition 4.4.2.6]). These results can be dualized, more or less straightforwardly.

2.6. **Comma ∞-categories.** The observation that the ∞-category $\widehat{\mathrm{Cat}}_\infty$ admits all limits allows us to define *over* and *under-∞-categories* in a model independent way, which is however essentially equivalent to Lurie's definition. We state their construction here, since over- and under-∞-categories will pop up often in the following.

Construction 2.6.1. Let $\mathscr{C}$ be any ∞-category, and let $\mathrm{Fun}\big(\Delta^1, \mathscr{C}\big)$ be the ∞-category of arrows in $\mathscr{C}$, with morphisms provided by homotopy coherent diagrams of arrows. The natural inclusions

$$\begin{pmatrix} s_0 \\ s_1 \end{pmatrix} \colon \{[0]\} \coprod \{[1]\} \hookrightarrow \Delta^1$$

induce by pre-composition two ∞-functors

$$\langle \mathrm{ev}_0, \mathrm{ev}_1 \rangle \colon \mathrm{Fun}\big(\Delta^1, \mathscr{C}\big) \longrightarrow \mathscr{C} \times \mathscr{C}$$

which determine the *source* and the *target* of a morphism, respectively. In particular, we can apply this construction to the ∞-category of homotopy coherent diagrams of shape K in $\mathscr{C}$ – i.e., to the ∞-category $\mathrm{Fun}(K, \mathscr{C})$: its arrow ∞-category $\mathrm{Fun}\big(\Delta^1, \mathrm{Fun}(K, \mathscr{C})\big)$ consists of homotopy coherent natural transformation of diagrams of shape K. In particular, we have an ∞-functor

$$\mathrm{ev}_1 \colon \mathrm{Fun}\big(\Delta^1, \mathrm{Fun}(K, \mathscr{C})\big) \longrightarrow \mathrm{Fun}(K, \mathscr{C}),$$

and given a fixed diagram $p \colon K \to \mathscr{C}$ we can take the fiber of such ∞-functor at p. This yields an ∞-category

$$\mathrm{Fun}(K, \mathscr{C})_{/p} := \mathrm{Fun}\big(\Delta^1, \mathrm{Fun}(K, \mathscr{C})\big) \times_{\mathrm{Fun}(K,\mathscr{C})} \{p\}\,.$$

Morally, this is the ∞-category of homotopy coherent natural transformations of diagrams of shape K, with the datum of a homotopy coherent natural equivalence between the target diagram and the diagram given by p. The composition

$$\mathrm{Fun}(K, \mathscr{C})_{/p} \longrightarrow \mathrm{Fun}\big(\Delta^1, \mathrm{Fun}(K, \mathscr{C})\big) \xrightarrow{\mathrm{ev}_0} \mathrm{Fun}(K, \mathscr{C})$$

selects the source of such natural transformation. Considering the constant diagram ∞-functor

$$\mathrm{const} \colon \mathscr{C} \longrightarrow \mathrm{Fun}(K, \mathscr{C}),$$

we produce a cospan of ∞-categories

$$\begin{tikzcd} & \mathscr{C} \arrow[d, "\mathrm{const}"] \\ \mathrm{Fun}\big(\Delta^1, \mathrm{Fun}(K, \mathscr{C})\big)_{/p} \arrow[r, "\mathrm{ev}_0"'] & \mathrm{Fun}(K, \mathscr{C}) \end{tikzcd}$$

and we can take its limit

$$\mathrm{Fun}(K, \mathscr{C})_{/p} := \mathrm{Fun}(K, \mathscr{C})_{/p} \times_{\mathrm{Fun}(K,\mathscr{C})} \mathscr{C}.$$

Again, this pullback amounts to the datum of a homotopy coherent natural transformation of diagrams of shape K, with the datum of a natural equivalence between the source diagram and the constant diagram over some object of $\mathscr{C}$, and the datum of a natural equivalence between the target diagram and the diagram p. This produces *precisely* the correct over-∞-category. The under-∞-category $\mathscr{C}_{p/}$ is defined analogously.

Remark 2.6.2. If $K \simeq \{*\}$, then a diagram of shape K corresponds to an object X in $\mathscr{C}$, and Construction 2.6.1 produces the slice ∞-categories $\mathscr{C}_{/X}$ and $\mathscr{C}_{X/}$.

2.7. **Fibrations and Grothendieck construction.** This section is arguably the most technical part of this foundational chapter, and this cannot be avoided: usually, graduate students do not encounter fibrations of categories during their studies, so they can fail to see what is happening when one introduces *fibrations* of ∞-categories – which appear in many flavors, many of which have no counterparts in the 1-categorical framework. We can try to explain their necessity as follows: in ordinary category theory, if one wants to define a functor $F \colon \mathscr{C} \to \mathscr{D}$ they can simply define the image of an object X in $\mathscr{C}$ under F, specify what is the image of a morphism $f : X \to Y$, and finally check that composition of arrows is preserved[4]. This *cannot hold* in ∞-category, and

[4]This is *beyond* wishful thinking: usually people stop at "defining the image of an object".

it is arguably the most crucially difficult part of the theory: since we have non-trivial n-cells for all $n \geqslant 2$ testifying to *how* diagrams of $(n-1)$-cells can commute, we should actually define for any n-simplex $\sigma\colon \Delta^n \to \mathscr{C}$ its image $F(\sigma)\colon \Delta^n \to \mathscr{D}$. Due to unfortunate reasons, we lack the time to specify an infinite amount of data.

Here is where fibrations enter the picture: they allow to control an *infinite* amount of homotopy coherencies via an only 1-dimensional check. To make this introduction less obscure, let us start with arguably *the* classical example of a fibration.

Example 2.7.1 (Categories fibered in groupoids and Grothendieck construction, [Sta23, Section 4.35]). Let $p\colon \mathscr{D} \to \mathscr{C}$ be a functor of categories.

Definition 2.7.2. We say that p exhibits $\mathscr{D}$ as *fibered in groupoids* over $\mathscr{C}$ if the following two conditions hold.

(1) For any arrow $f : X \to Y$ in $\mathscr{C}$ and any lift of the target Y to $\mathscr{D}$ there exists a complete lift of f to $\mathscr{D}$, i.e., for all objects V in $\mathscr{D}$ such that $p(V) = Y$ we can find an arrow $\varphi\colon U \to V$ such that $F(\varphi) = f$.
(2) All arrows in $\mathscr{D}$ are p-*Cartesian*. In general, an arrow $\varphi\colon U \to V$ in $\mathscr{D}$ is said to be p-Cartesian if for every diagram of solid arrows in $\mathscr{D}$

$$\begin{tikzcd} & W \arrow[d, "\psi"] \\ U \arrow[r, "\varphi"'] & V \end{tikzcd}$$

and for every arrow $h\colon p(U) \to P(W)$ making the following diagram commutative in $\mathscr{C}$

$$\begin{tikzcd} & p(W) \arrow[dl, dotted, bend right, "h"'] \arrow[d, "p(\psi)"] \\ p(U) \arrow[r, "p(\varphi)"'] & p(V) \end{tikzcd}$$

then there exists a *unique* lift $\chi : U \to W$ such that $p(\chi) = h$ and such that the following diagram

$$\begin{tikzcd} & W \arrow[dl, dotted, bend right, "\chi"'] \arrow[d, "\psi"] \\ U \arrow[r, "\varphi"'] & V \end{tikzcd}$$

is commutative in $\mathscr{D}$.

It is a historical result that these data indeed define a *fibration in groupoids*, i.e., for any object X in $\mathscr{C}$ the fiber category

$$\mathscr{D}_X := \mathscr{D} \times_{\mathscr{C}} \{X\}$$

is actually a groupoid. What is more, the association

$$X \mapsto \mathscr{D}_X$$
$$\{f : X \to Y\} \mapsto \{f^* \colon \mathscr{D}_Y \to \mathscr{D}_X\},$$

is (weakly) 2-functorial. Here, f^* is defined by sending an object U in the fiber $\mathscr{D}_Y$ to the domain of the Cartesian lift of f having U as target (this exists in virtue of Definition 2.7.2.(1)), and by sending an arrow $h\colon V \to U$ in $\mathscr{D}_Y$ to the dotted arrow between lifts in $\mathscr{D}_X$ (which exists in virtue of Definition 2.7.2.(2)). Moreover, the uniqueness of such lift provided by Definition 2.7.2.(2) implies that there is a natural 2-cell that testifies to the fact that $(g \circ f)^* \simeq f^* \circ g^*$. In particular, giving a fibration in groupoids $p\colon \mathscr{D} \to \mathscr{C}$ is equivalent to giving a weak 2-functor from $\mathscr{C}^{\mathrm{op}}$ to the $(2,1)$-category of groupoids Gpd.

Remark 2.7.3. In the above example, one can relax Definition 2.7.2 in the following way: for any arrow $f : X \to p(V)$, we can simply ask for the existence of *some* p-Cartesian morphism $U \to V$ lifting f. This is enough to provide a weak 2-functor $\mathscr{C}^{\mathrm{op}} \to \mathrm{Cat}$, but not enough to prove that every fiber is a groupoid. This construction is known in literature as the *Grothendieck construction*.

Notice that, in the above example, a clever definition of the 1-dimensional data guarantees a canonical 2-homotopy between the composition of pullback functors and the pullback functor of the composition: this seems a particularly useful strategy when dealing with ∞-categories.

Definition 2.7.4 (Fibrations of ∞-categories, [Lur09, Chapter 2])**.** Left $p\colon \mathscr{D} \to \mathscr{C}$ be an ∞-functor.

(1) We say that p is a *inner fibration* if it has the right lifting property with respect to inner horn inclusions $\Lambda^n_k \subseteq \Delta^n$ for any n and any $0 < k < n$.
(2) If $p\colon \mathscr{D} \to \mathscr{C}$ is an inner fibration of ∞-categories, we say that an arrow $f : \Delta^1 \to \mathscr{D}$ in $\mathscr{D}$ is *p-cartesian* if the natural ∞-functor between over-∞-categories

$$\mathscr{D}_{/f} \longrightarrow \mathscr{D}_{/\mathrm{ev}_1(f)} \times_{\mathscr{C}_{/\mathrm{ev}_1(p\circ f)}} \times \mathscr{C}_{/p\circ f}$$

is a trivial Kan fibration. We say that p is a *Cartesian fibration* if it is an inner fibration and for any morphism $f : X \to p(U)$ in $\mathscr{C}$ there is a p-Cartesian morphism $V \to U$ in $\mathscr{D}$ lifting f.
(3) We say that p is a *coCartesian fibration* if $p^{\mathrm{op}}\colon \mathscr{D}^{\mathrm{op}} \to \mathscr{C}^{\mathrm{op}}$ is a Cartesian fibration.
(4) We say that p is a *left fibration* if it has the right lifting property with respect to inner and outer-left horn inclusions $\Lambda^n_k \subseteq \Delta^n$ for any n and any $0 \leqslant k < n$.
(5) We say that p is a *right fibration* if it has the right lifting property with respect to inner and outer-right horn inclusions $\Lambda^n_k \subseteq \Delta^n$ for any n and any $0 < k \leqslant n$.
(6) We say that p is a *Kan fibration* if it has the right lifting property with respect to *all* horn inclusions $\Lambda^n_k \subseteq \Delta^n$ for any n and any $0 \leqslant k \leqslant n$.
(7) We say that p is a *trivial Kan fibration* if it has the right lifting property with respect to boundary inclusions $\partial\Delta^n \subseteq \Delta^n$ for any n.

Remark 2.7.5. Following the usual principle of these lectures – that is, *simplicial sets are just a model* – we have introduced only those fibrations of ∞-categories that have both an intrinsic categorical meaning and whose definition can be straightforwardly interpreted in the internal theory of ∞-categories, while keeping an essentially combinatorial and simplicial flavour. Namely:

(1) Inner fibrations can be thought as families of ∞-categories $\mathscr{D}_X$ parametrized by objects of $\mathscr{C}$, such that for any morphism $f: X \to Y$ there is a *correspondence* $\mathscr{D}_X^{\mathrm{op}} \times \mathscr{D}_Y \to \mathscr{S}$ between $\mathscr{D}_X$ and $\mathscr{D}_Y$. This is spelled out in some detail in [Lur09, Section 2.3.1].

(2) Kan fibrations were already defined in terms of fibrations for the model category of simplicial sets, but here they acquire further meaning: they can be thought of as families of ∞-*groupoids* $\mathscr{D}_X$ parametrized by objects of $\mathscr{C}$, such that for any morphism $f: X \to Y$ in $\mathscr{C}$ there is a couple of adjoint ∞-functors

$$f^*: \mathscr{D}_Y \rightleftarrows \mathscr{D}_X : f_*.$$

When a Kan fibration is trivial in the sense of Definition 2.7.4.(7), then the fibration $p: \mathscr{D} \to \mathscr{C}$ it is moreover an equivalence of ∞-categories.

(3) Cartesian fibrations, unsurprisingly conceptually but strikingly technically, parametrize contravariant ∞-functors $\mathscr{C}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$, while coCartesian fibrations parametrize covariant ∞-functors $\mathscr{C} \to \widehat{\mathrm{Cat}}_\infty$. We shall review this correspondence in Theorem 2.7.13.

(4) Right fibrations are nothing more than Cartesian fibrations fibered in ∞-groupoids, i.e., contravariant ∞-functors $\mathscr{C}^{\mathrm{op}} \to \mathscr{S}$ ([Lur09, Proposition 2.4.2.4]). Similarly, left fibrations are coCartesian fibrations fibered in ∞-groupoids, hence covariant ∞-functors $\mathscr{C} \to \mathscr{S}$. Again, we shall review in more detail this correspondence in Theorem 2.7.7.

The order of the definitions in Definition 2.7.4 is chosen appropriately: any class of fibrations contains *all* the classes of fibrations appearing after that. Moreover, the conceptual description of fibrations just provided makes believable the following claims: if $\mathscr{C}$ is a ∞-groupoid then any left or right fibration is automatically a Kan fibration ([Lur09, Proposition 2.1.3.3]); if moreover $\mathscr{D}$ is an ∞-groupoid as well, then every Cartesian fibration $p: \mathscr{C} \to \mathscr{D}$ is also a right fibration, while every coCartesian fibration is also a left fibration.

One of the most fundamental results in ∞-category theory is the fact that Cartesian and coCartesian fibrations are indeed a way to encode $(\infty, 2)$-functors with values in Cat_∞, and that right and left fibrations encode $(\infty, 2)$-functors with values in $\mathscr{S}$. This equivalence is provided by a Quillen equivalence of model categories, in the following way.

Construction 2.7.6 (Preliminary notations for the Grothendieck construction)**.**

(1) Let $\mathscr{C}$ be any simplicially enriched category, and let sSet be the category of simplicial sets endowed with the Quillen model structure of Theorem 1.3.9. The category of functors $\mathrm{sSet}^{\mathscr{C}}$ carries a projective model structure which detects weak equivalences and fibrations point-wise.

(2) Let S be any simplicial set. The over category $\mathrm{sSet}_{/S}$ carries a contravariant model structure which detects cofibrations at the level of the underlying map of simplicial sets (i.e., they are monomorphisms) and for which weak equivalences are maps $X \to Y$ such

that

$$X^{\triangleright}\coprod_X S \longrightarrow Y^{\triangleright}\coprod_Y S$$

is a weak equivalence for the Joyal model structure on simplicial sets of Theorem 2.3.6. Here, the notation $X^{\triangleright}$ denotes the join simplicial set $X \star [0]$: informally, we are adding an artificial final vertex, the *cone pont*, to X. This model structure enjoys a vast amount of nice technical and theoretical properties (it is proper, simplicial and combinatorial), and the fibrant and cofibrant objects are precisely right fibrations $X \to S$.

(3) Let $\varphi\colon \mathfrak{C}[S] \to \mathscr{C}^{\mathrm{op}}$ be a simplicial functor, where $\mathfrak{C}[S]$ is the simplicially enriched category associated to S via the functor 2.1.17. For any object $X \to S$ in $\mathrm{sSet}_{/S}$, by the functoriality of $\mathfrak{C}$, we obtain both $\mathfrak{C}[X] \to \mathfrak{C}[X^{\triangleright}]$ and $\mathfrak{C}[X] \to \mathfrak{C}[S] \to \mathscr{C}^{\mathrm{op}}$, hence we can consider the pushout of simplicially enriched categories

$$\mathscr{M} := \mathfrak{C}[X^{\triangleright}]\coprod_{\mathfrak{C}[X]}\mathscr{C}^{\mathrm{op}}.$$

If v denotes the cone point of $X^{\triangleright}$, then $\mathscr{M}$ trivially contains a copy of both v and $\mathscr{C}^{\mathrm{op}}$, and so we can define a *covariant* functor

$$\mathrm{St}_{\varphi}X : \mathscr{C} \longrightarrow \mathrm{sSet},$$

defined on an object C of $\mathscr{C}$ as the mapping complex

$$\mathrm{St}_{\varphi}X(C) := \mathrm{Map}_{\mathscr{M}}(C, v).$$

Varying all over objects X's over S, we define a functor

$$\mathrm{St}_{\varphi} : \mathrm{sSet}_{/S} \longrightarrow \mathrm{sSet}^{\mathscr{C}}$$

that we call the *straightening functor associated to* φ. Suppose now that $\mathscr{C} = \mathfrak{C}[S]$ and $\varphi = \mathrm{id}_{\mathfrak{C}[S]}$. For $X \to S$ a morphism, we can write the simplicially enriched category

$$\mathscr{M} := \mathfrak{C}[X^{\triangleright}]\coprod_{\mathfrak{C}[X]}\mathfrak{C}[S] \cong \mathfrak{C}\left[X^{\triangleright}\coprod_X S\right]$$

because $\mathfrak{C}$ is a left Quillen functor, hence preserves colimits. This is the free simplicially enriched category over $X^{\triangleright}\coprod_X S$, which is simply S endowed with an extra vertex v and a set of n-simplices which remember the existence of the $(n-1)$-simplices in X. For example, let $X = \Delta^1$ and $S = \{*\}$. Then $X^{\triangleright} \cong \Delta^2$, where the point v plays the role of $[2]$, and $X^{\triangleright}\coprod_X S$ is just the simplicial set obtained by Δ^2 collapsing the 1-simplex $[0] \to [1]$ to a point, *without imposing further relations on* $f : [0] \to [2]$ *and on* $g : [1] \to [2]$, or on any other higher dimensional simplex for that matter.

$$X^{\triangleright}\coprod_X S \cong \left\{ \begin{tikzcd} {[0]\equiv[1]} \arrow[d, bend right, "g"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "\Rightarrow"] \\ v \end{tikzcd} \right\}.$$

So we can compute a bit more easily the above mapping complex: for an object C in $\mathfrak{C}[S]$ (which corresponds to a vertex of S), the set of vertices of the mapping complex

$\mathrm{Map}_{\mathcal{M}}(C, v)$ is the set of maps $C \to v$, which in turn corresponds to the set of vertices in the fiber of C under the map $X \to S$, because for any object x in the fiber of C, there is a unique map $x \to v$. So we do recover the idea of constructing a functor that sends an object to its fiber.

(4) The straightening functor admits a right adjoint

$$\mathrm{Unst}_{\varphi} \colon \mathrm{sSet}^{\mathscr{C}} \longrightarrow \mathrm{sSet}_{/S},$$

whose explicit description is provided by Rezk in [Rez].

Theorem 2.7.7 (∞-categorical Grothendieck construction, [Lur09, Theorem 2.2.1.2])**.** *The adjunction*

$$\mathrm{St}_{\varphi} \colon \mathrm{sSet}_{/S} \rightleftarrows \mathrm{sSet}^{\mathscr{C}} \colon \mathrm{Unst}_{\varphi}$$

is a Quillen adjunction, and if φ is an equivalence of simplicially enriched categories then such adjunction is an equivalence. In particular, the ∞-category of left fibrations over an ∞-category $\mathscr{C}$ is equivalent to the ∞-category of ∞-functors $\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathcal{S})$.

Outline of the proof. A messy, yet straightforward, computation yields that St_{φ} indeed does preserve both cofibrations and weak equivalences, hence the adjunction $\mathrm{St}_{\varphi} \dashv \mathrm{Unst}_{\varphi}$ is Quillen. To prove that it is an equivalence, one proceeds as follows.

(1) First, one proves that indeed right fibrations are sufficient to capture the homotopy theory of $\mathrm{sSet}_{/S}$ (i.e., the inclusion of the full simplicially enriched sub-category $\mathrm{RFib}(S)$ spanned by right fibrations over S inside $\mathrm{sSet}_{/S}$ is an equivalence of simplicially enriched categories): this is [Lur09, Lemma 2.2.3.9].

(2) Next, assume that $S = \Delta^n$. In this case, $\mathfrak{C}[\Delta^n]$ is a simplicially enriched refinement of the usual poset category $[n]$: it has the same objects, but the condition that the composition of two morphisms $i \to j$ and $j \to k$ is *strictly equal* to the morphism $i \to k$ is relaxed to the existence of a coherent homotopy between the two morphisms. In more technical terms, the (geometric realization of the) mapping complex $\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i, j)$ is homeomorphic to a cube (with interior, hence contractible), whose vertices are all the possible compositions $i \to k_1 \to k_2 \to \ldots \to k_n = j$. In particular, we can consider the map $\varphi \colon \mathfrak{C}[\Delta^n] \to [n]$ given by the identity on objects (we are "collapsing" our coherent homotopies to strict equalities). Then one sees that the unstraightening functor induces an equivalence between (the simplicially enriched categories associated to) fibrant and cofibrant objects in sSet^{Δ^n} and $\mathrm{RFib}(\Delta^n)$ by directly inspecting the counit natural transformation induced between their homotopy categories [Lur09, Lemmas 2.2.3.1 and 2.2.3.10]).

(3) One proves that if $\mathcal{U}$ is any class of simplicial sets which contains Δ^n for any n, which is stable under isomorphisms, disjoint unions, and pushouts and filtered colimits along monomorphisms, then $\mathcal{U} = \mathrm{sSet}$ ([Lur09, Lemma 2.2.3.5]). In particular, one proves that the class $\mathcal{U}$ of simplicial sets for which the unstraightening functor produces an equivalence of simplicially enriched categories between fibrant and cofibrant objects in sSet^S and $\mathrm{RFib}(S)$ enjoys all these properties ([Lur09, Lemma 2.2.3.11]).

(4) Finally, one proves that if $\mathscr{C} \to \mathscr{C}'$ is an equivalence of simplicially enriched categories then $\mathrm{sSet}^{\mathscr{C}} \simeq \mathrm{sSet}^{\mathscr{C}'}$ ([Lur09, Proposition A.3.3.8]), and that a Quillen adjunction is a Quillen equivalence if and only if it induces an equivalence between the associated simplicially enriched categories of fibrant and cofibrant objects ([Lur09, Proposition A.3.1.10]).

Therefore, the steps above together imply the whole statement. □

Theorem 2.7.7 is the higher categorical generalization of Grothendieck's fibrations in groupoids. Yet, we still need an analogue result relating Cartesian fibrations and contravariant ∞-functors with values in Cat_∞. The proof of this statement is *way* more technical, and relies on introducing a new model category.

Definition 2.7.8 (Marked simplicial sets, [Lur09, Definition 3.1.0.1])**.** A *marked simplicial set* is a pair $(S, \mathscr{E})$ where S is a simplicial set and $\mathscr{E}$ is a set of edges which contain every degenerate edge of S. An edge belonging to $\mathscr{E}$ will be a *marked* edge.
A map of marked simplicial sets $(S, \mathscr{E}) \to (S', \mathscr{E}')$ is a map of simplicial sets which sends marked edges to marked edges. The category of marked simplicial sets is denoted by sSet^+.

The meaning of *marking* a simplicial set is the following: we want to specify *a priori* what should be assumed to be an equivalence in an ∞-category modeled by a quasi-category. Of course, every identity morphism has to be an equivalence, hence the assumption on degenerate edges.

Remark 2.7.9. The category sSet^+ is monoidal Cartesian closed: for any two objects $(X, \mathscr{E})$, $(Y, \mathscr{F})$, there exists an internal marked mapping complex

$$\underline{\mathrm{Map}}^+((X,\mathscr{E}),(Y,\mathscr{F}))$$

which provides a right adjoint to the Cartesian product of marked simplicial sets. We shall denote by

$$\mathrm{Map}^+((X,\mathscr{E}),(Y,\mathscr{F})) \subseteq \underline{\mathrm{Map}}^+((X,\mathscr{E}),(Y,\mathscr{F}))$$

the maximal marked sub-simplicial set of $\underline{\mathrm{Map}}^+((X,\mathscr{E}),(Y,\mathscr{F}))$ which comprises only those simplices whose edges are all marked in $\underline{\mathrm{Map}}^+((X,\mathscr{E}),(Y,\mathscr{F}))$. For any simplicial set S, we can consider the category $\mathrm{sSet}^+_{/S^\sharp}$: here, we shall denote by

$$\underline{\mathrm{Map}}^+_S((X,\mathscr{E}),(Y,\mathscr{F})) \subseteq \underline{\mathrm{Map}}^+((X,\mathscr{E}),(Y,\mathscr{F}))$$

and by

$$\mathrm{Map}^+_S((X,\mathscr{E}),(Y,\mathscr{F})) \subseteq \mathrm{Map}^+((X,\mathscr{E}),(Y,\mathscr{F}))$$

the sub-simplicial sets spanned by those edges compatible with the maps toward S.

Construction 2.7.10. For any simplicial set X we have three notable examples of markings.

(1) The marked simplicial set $X^\sharp$ is the maximally marked simplicial set (all edges are marked).

(2) The marked simplicial set $X^\flat$ is the minimally marked simplicial set (only degenerate edges are marked).

(3) If X is endowed of a map $p: X \to S$, the marked simplicial set $X^\natural$ is the Cartesian marked simplicial set (marked edges are p-Cartesian edges, in the sense of Definition 2.7.4.(2)).

Proposition 2.7.11 (Cartesian model structure on marked simplicial sets, [Lur09, Proposition 3.1.3.7])**.** *Let S be any simplicial set. The category $\mathrm{sSet}^+_{/S^\sharp}$ of marked simplicial sets over S is endowed with a left proper combinatorial and simplicial model structure, called the Cartesian model structure, described as follows. Cofibrations are the ones whose underlying morphism of simplicial sets is a cofibration for the Quillen model structure on simplicial sets. Weak equivalences are maps $f: X \to Y$ such that, for any Cartesian fibration $p: Z \to S$, pre-composition with f induces equivalences*

$$\underline{\mathrm{Map}}^+_S(Y, Z^\natural) \simeq \underline{\mathrm{Map}}^+_S(X, Z^\natural)$$

and[5]

$$\mathrm{Map}^+_S(Y, Z^\natural) \simeq \mathrm{Map}^+_S(X, Z^\natural).$$

Fibrant and cofibrant objects are precisely Cartesian fibrations over S.

Remark 2.7.12. If $S = \{*\}$, then the ∞-category associated to such a model structure is again Cat_∞, and the Quillen inclusion $(-)^\sharp: \mathrm{sSet} \subseteq \mathrm{sSet}^+$ which marks every edge models the natural inclusion $\mathcal{S} \subseteq \mathrm{Cat}_\infty$. Differently from the Joyal model structure on simplicial sets, however, the Cartesian model structure on marked simplicial sets enjoys nicer technical properties. For example, the fact that this is a *simplicial* model category provides a somewhat "more obvious" hsSet-enrichment: indeed, the mapping complex $\underline{\mathrm{Map}}^+$ introduced in Remark 2.7.9 models the *internal* mapping object in Cat_∞ – i.e., the ∞-category of ∞-functors – while its subcomplex Map^+ is the maximal sub-∞-groupoid that provides the natural $\mathcal{S}$-enrichment. This will play a role in describing how Cat_∞ is a *presentable* ∞-category in Section 2.9.

Theorem 2.7.7 refines to the following "marked" enhancement, which is a more technically involved rephrasing of the previous statement (and of the strategy of its proof, as well).

Theorem 2.7.13 (Marked ∞-categorical Grothendieck construction, [Lur09, Theorem 3.2.0.1])**.** *For any simplicial set S and for every simplicial functor $\varphi: \mathfrak{C}[S] \to \mathscr{C}$ of simplicially enriched categories there exists a Quillen adjunction*

$$\mathrm{St}_\varphi : \mathrm{sSet}^+_{/S^\sharp} \rightleftarrows \left(\mathrm{sSet}^+\right)^{\mathscr{C}} : \mathrm{Unst}_\varphi$$

where the source is endowed with the Cartesian model structure of Proposition 2.7.11 and the target again with the projective model structure. If φ is an equivalence of simplicially enriched categories, then this adjunction is a Quillen equivalence.

Remark 2.7.14. Until now, it is not clear how the Cartesianity condition makes it easier to construct ∞-functors with values in homotopy types or ∞-categories. [Lur09, Remark 2.4.1.4] makes this advantage precise: if $p: \mathscr{D} \to \mathscr{C}$ is an inner fibration, then an arrow $f: \Delta^1 \to \mathscr{D}$ is

[5] Actually, either of the two implies the other.

p-cartesian precisely if for any $n \geqslant 2$ and for every diagram of solid ∞-functors

$$\begin{tikzcd} \Delta^1 \arrow[d, hook, "{\{n-1,n\}}"'] \arrow[drr, bend left, "f"] & & \\ \Lambda^n_n \arrow[rr] \arrow[d, hook] & & \mathscr{D} \arrow[d, "p"] \\ \Delta^n \arrow[rr] \arrow[urr, dotted] & & \mathscr{C} \end{tikzcd}$$

there exists a lift $\Delta^n \to \mathscr{D}$ making everything commute.

Example 2.7.15 (Notable example of fibrations).

(1) Let $\mathscr{C}$ be any ∞-category, and let $\mathrm{Fun}(\Delta^1, \mathscr{C})$ be the ∞-category of arrows in $\mathscr{C}$. The source evaluation ∞-functor

$$\mathrm{ev}_0 \colon \mathrm{Fun}(\Delta^1, \mathscr{C}) \longrightarrow \mathscr{C}$$

is a right fibration, while the target evaluation ∞-functor

$$\mathrm{ev}_1 \colon \mathrm{Fun}(\Delta^1, \mathscr{C}) \longrightarrow \mathscr{C}$$

is a left fibration, and moreover these left/right fibrations are compatible one with the other. The ∞-functor

$$\langle \mathrm{ev}_0, \mathrm{ev}_1 \rangle \colon \mathrm{Fun}(\Delta^1, \mathscr{C}) \longrightarrow \mathscr{C} \times \mathscr{C}$$

is, in fact, a *bifibration* ([Lur09, Corollary 2.4.7.11]): we shall not review this definition in these lectures since this is the only instance in which we really use it, but it amounts to little more than what we said here. In any case, Theorem 2.7.7 guarantees that we have a well-defined ∞-functor

$$\mathrm{Map}_{\mathscr{C}}(-,-) \colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \longrightarrow \mathscr{S}.$$

(2) Recall our Construction 2.6.1 of comma ∞-categories. For any diagram $p \colon K \to \mathscr{C}$ in $\mathscr{C}$, the natural ∞-functor

$$\mathscr{C}_{/p} \longrightarrow \mathscr{C}$$

is a left fibration ([Lur09, Proposition 2.1.2.1]). If $p \colon \{*\} \to \mathscr{C}$ selects the final object, then it is a trivial Kan fibration (this is actually how final objects are defined in [Lur09]).

(3) An ∞-functor $F \colon \mathscr{C} \to \mathscr{D}$ can be described as an arrow in $\widehat{\mathrm{Cat}}_\infty$, hence as an ∞-functor $F \colon \Delta^1 \to \widehat{\mathrm{Cat}}_\infty$. By Theorem 2.7.13 (and by suitably enlarging our universe), we can view such an ∞-functor as a coCartesian fibration $p \colon \mathscr{M} \to \Delta^1$, where $\mathscr{M} \times_{\Delta^1} \{[0]\} \simeq \mathscr{C}$ and $\mathscr{M} \times_{\Delta^1} \{[1]\} \simeq \mathscr{D}$. If this fibration is also Cartesian, then it defines an ∞-functor $(\Delta^1)^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ which again sends $[0]$ to $\mathscr{C}$ and $[1]$ to $\mathscr{D}$, i.e., it also defines an ∞-functor $G \colon \mathscr{D} \to \mathscr{C}$. By unraveling the definitions of p-coCartesian and p-Cartesian edges, we see that they are equivalent to saying that $F \dashv G$. This is how [Lur09, Definition 5.2.2.1] defines adjunctions in the ∞-categorical setting.

2.8. **The ∞-categorical Yoneda Lemma.** At this point, we have all the needed ingredients to produce an ∞-categorical statement of the (Ninja) Yoneda Lemma (Theorem 1.4.1).

(1) We have a well defined ∞-category of ∞-functors $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ for any couple of ∞-categories $\mathscr{C}$ and $\mathscr{D}$.
(2) We have a well defined opposite ∞-category $\mathscr{C}^{\mathrm{op}}$.
(3) Since any ∞-category is canonically enriched over homotopy types, it is a tame guess to imagine that we should replace the category of sets with the ∞-category of spaces $\mathscr{S}$.
(4) Moreover, Example 2.7.15.(1) (together with Theorem 2.7.13) guarantees that we have a well defined ∞-functor

$$\mathrm{Map}_{\mathscr{C}}(-,-)\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \longrightarrow \mathscr{S},$$

hence by passing through the adjoint

$$よ\colon \mathscr{C} \longrightarrow \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{S})$$

we have a good candidate for the ∞-categorical Yoneda embedding.

This construction should meet the following expectations: the ∞-functor

$$よ\colon \mathscr{C} \longrightarrow \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{S}),$$

should be a fully faithful ∞-functor, which should preserve all limits existing in $\mathscr{C}$, and should produce a cocomplete ∞-category of presheaves. Finally, we expect that restricting along よ should provide, for any cocomplete ∞-category $\mathscr{D}$, an equivalence of ∞-categories of ∞-functors

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{S}), \mathscr{D}) \xrightarrow{\simeq} \mathrm{Fun}(\mathscr{C}, \mathscr{D}) \tag{2.8.1}$$

where in the left hand side $\mathrm{Fun}^{\mathrm{L}}$ denotes the ∞-category of ∞-functors which preserve colimits[6]. The good news is: everything stated above is true, and it is even available in the literature[7].

Theorem 2.8.2 (∞-categorical Yoneda Lemma, [Lur09, Propositions 5.1.3.1, 5.1.3.2, Corollary 5.1.2.4, Theorem 5.1.5.6])**.** *For any ∞-category $\mathscr{C}$, there exists an ∞-categorical Yoneda embedding*

$$よ\colon \mathscr{C} \hookrightarrow \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{S}),$$

informally described by the association $C \mapsto \left\{\mathrm{Map}_{\mathscr{C}}(-, C)\colon \mathscr{C}^{\mathrm{op}} \to \mathscr{S}\right\}$, which is fully faithful and preserves all limits existing in $\mathscr{C}$. Moreover, よ exhibits $\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{S})$ as the free cocompletion of the ∞-category $\mathscr{C}$.

Remark 2.8.3. The statement of Theorem 2.8.2 can be refined as follows. The association $\mathscr{C} \mapsto \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{S})$ can be made functorial, yielding an ∞-functor

$$\mathscr{P}\colon \mathrm{Cat}_\infty \longrightarrow \widehat{\mathrm{Cat}}_\infty,$$

which sends a ∞-functor $F\colon \mathscr{C} \to \mathscr{D}$ to the left Kan extension of F along the Yoneda embedding よ$\colon \mathscr{C} \hookrightarrow \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{S})$. Moreover, there exists a natural transformation of ∞-functors from

[6] The L in $\mathrm{Fun}^{\mathrm{L}}$ stands for *left*, since under very mild assumption colimit preserving ∞-functors are precisely left adjoints, see Theorem 2.9.3.

[7] This last part is not something to be taken for granted, not even in the case of "obviously true" statements.

Cat_∞ to $\widehat{\mathrm{Cat}}_\infty$ between the natural inclusion and the ∞-functor $\mathscr{P}$, which agrees with the ∞-categorical Yoneda embedding. This highly technical result has been recently proved in [HHLN20, Theorem 8.1] and refined to the enriched setting in the last months in [Mos23, Theorem 3.6], taking into account the adjunction-like behavior of the equivalence 2.8.1 as well. This will not used in any essential way in the following.

2.9. **Presentable ∞-categories.** A key role in ∞-category theory is played by those ∞-categories that are *presentable*. In the classical setting, a (locally) presentable category $\mathscr{C}$ is a, possibly large, cocomplete category which is however controlled by a small set of compact objects (called the *generators* of $\mathscr{C}$) under κ-filtered colimits, for some regular cardinal κ. Since we know that all these ingredients are available in the ∞-categorical theory, we can just cast this definition word by word also for ∞-categories.

Definition 2.9.1 ([Lur09, Definition 5.5.0.1])**.** An ∞-category $\mathscr{C}$ is *presentable* if it satisfies the following two conditions.

(1) It is cocomplete.
(2) It is κ-accessible for some regular cardinal κ, i.e., there exists a small category $\mathscr{C}^\omega$ and a regular cardinal κ such that

$$\mathscr{C} \simeq \mathrm{Ind}_\kappa(\mathscr{C}^\omega),$$

where $\mathrm{Ind}_\kappa(\mathscr{C}^\omega)$ is the ∞-category of κ-*filtered diagrams on* $\mathscr{C}^\omega$ obtained by formally adjoining all colimits of κ-filtered diagrams in $\mathscr{C}^\omega$.

Definition 2.9.1 already gives a glimpse of the convenience of the existence of such small ∞-category of compact objects. But presentable ∞-categories are *way more convenient than that*, as the following result (which is a ∞-categorical rephrasing of theorems by Simpson and Dugger) shows.

Theorem 2.9.2 ([Lur09, Theorems 5.5.1.1 and Proposition A.3.7.6])**.** *For $\mathscr{C}$ an ∞-category, the following are equivalent.*

(1) The ∞-category $\mathscr{C}$ is presentable.
(2) The ∞-category $\mathscr{C}$ is locally small, admits all colimits, and there exist a regular cardinal κ and a small set S of κ-compact generators such that every object in $\mathscr{C}$ can be written as a small colimit over a diagram in the full sub-∞-category spanned by objects in S.
(3) The ∞-category $\mathscr{C}$ arises as an accessible localization of the ∞-category of presheaves $\mathrm{Fun}(\mathscr{D}, \mathscr{S})$ *over a small ∞-category $\mathscr{D}$, i.e., it is an accessible full sub-∞-category closed under all limits in* $\mathrm{Fun}(\mathscr{D}, \mathscr{S})$ *and the inclusion admits a left adjoint.*
(4) The ∞-category $\mathscr{C}$, in its incarnation as a quasi-category, is equivalent to the simplicial nerve of the Dwyer-Kan localization of $\mathscr{A}_{\mathrm{cf}}$, where $\mathscr{A}$ is some simplicial combinatorial model category.

In fact, presentable ∞-categories are *so* nice that they provide a wide class of ∞-categories where the converse to the statements*left adjoints preserve colimits* and *right adjoints preserve limits* holds.

Theorem 2.9.3 (Adjoint Functor Theorem, [Lur09, Corollary 5.5.2.9]). *Let $F : \mathscr{C} \to \mathscr{D}$ be an ∞-functor between presentable ∞-categories.*

(1) The ∞-functor F admits a right adjoint if and only if it preserves small colimits.
(2) The ∞-functor F admits a left adjoint if and only if it preserves small limits and κ-filtered colimits for some regular cardinal κ.

It is natural to ask for the existence of an ∞-category Pr^{L} of presentable ∞-categories and left adjoints between them; equivalently, in virtue of 2.9.3, we can ask for an ∞-category Pr^{R} of presentable ∞-categories and right adjoints between them. Moreover, they should be equivalent via an equivalence that fixes objects and reverts arrows, informally by selecting a right or left adjoint, respectively. Indeed, this can be done: we can always build a sub-∞-category $\mathscr{C}$ out of another ∞-category $\mathscr{D}$ by only specifying what kind of objects and arrows of $\mathscr{D}$ we want our sub-∞-category $\mathscr{C}$ to contain.

Theorem 2.9.4 ([Lur09, Chapter 5]). *Let $\widehat{\mathrm{Cat}}_\infty^{\mathrm{rex}}$ be the sub-∞-category of $\widehat{\mathrm{Cat}}_\infty$ spanned by cocomplete ∞-categories with colimit preserving ∞-functors. There exists a full sub-∞-category $\mathrm{Pr}^{\mathrm{L}} \subseteq \widehat{\mathrm{Cat}}_\infty$ spanned by presentable ∞-categories, and the inclusions*

$$\mathrm{Pr}^{\mathrm{L}} \subseteq \widehat{\mathrm{Cat}}_\infty$$

preserve filtered colimits and all limits. Moreover, there exists a complete ∞-category Pr^{R} of presentable ∞-categories and right adjoints between them, together with an anti-equivalence

$$\left(\mathrm{Pr}^{\mathrm{L}}\right)^{\mathrm{op}} \simeq \mathrm{Pr}^{\mathrm{R}}$$

which is the identity on objects. The inclusion $\mathrm{Pr}^{\mathrm{R}} \subseteq \widehat{\mathrm{Cat}}_\infty$ preserves all limits.

Theorem 2.9.4 gives *a lot* of information on Pr^{L}. It says that it is a complete and cocomplete ∞-category, whose limits and filtered colimits are computed as in $\widehat{\mathrm{Cat}}_\infty$. Moreover, arbitrary colimits are computed by passing through the diagram of right adjoints (which is now a diagram in Pr^{R}) and computing the *limit* on the underlying diagram of ∞-categories there. In particular, Pr^{L} is a *semi-additive ∞-category* (products and coproducts in Pr^{L} coincide).

Example 2.9.5 (Notable example of presentable ∞-categories).

(1) The ∞-category of spaces $\mathcal{S}$ is presentable. Every (small) space X is actually a colimit of a diagram of constant value in the point $\{*\}$. This formalizes the idea that the homotopy theory of topological spaces is completely modeled by CW complexes, which are obtained by gluing cells homeomorphic to $\mathbb{R}^n$ (hence, they are contractible and homotopy equivalent to a point), and every CW complex is a filtered colimit of its finite sub-CW-complexes.
(2) The ∞-category of (small) ∞-categories Cat_∞ is presentable.
(3) The ∞-category of presentable ∞-categories Pr^{L} is *not* presentable. However, fixing a regular cardinal κ, the ∞-category $\mathrm{Pr}^{\mathrm{L}}_\kappa$ of presentable ∞-categories which are λ-accessible for any regular cardinal λ up to κ is accessible ([Lur17, Lemma 5.3.2.9]).
(4) If $\mathscr{C}$ is a presentable ∞-category, then the ∞-category $\mathrm{Fun}(K, \mathscr{C})$ of homotopy coherent diagrams of shape K is presentable for any small ∞-category K ([Lur09, Proposition 5.5.3.6]).

(5) In **??**, we shall see that all our ∞-categories of algebraic interest are presentable, including modules and algebras over ordinary commutative rings.

We conclude this section, and this chapter, with an important subclass of presentable ∞-categories.

Definition 2.9.6 ([Lur09, Definition 6.1.0.4])**.** An ∞-category is an ∞-*topos* if it is an accessible left exact localization of an ∞-category of presheaves over a small ∞-category $\mathscr{C}$.

Recall that a localization $\iota\colon \mathscr{C} \hookrightarrow \mathscr{D}$ is left exact if the left adjoint reflector $\mathrm{L}\colon \mathscr{D} \to \mathscr{C}$ also preserves limits. In particular, in virtue of the characterization of presentable ∞-categories of Theorem 2.9.2.(4), an ∞-topos is a presentable ∞-category with this further requirement on the reflector ∞-functor.

Example 2.9.7 (Notable examples of ∞-topoi)**.**

(1) The ∞-category of spaces $\mathcal{S}$ is the prototypical example of an ∞-topos.
(2) Any ∞-category of presheaves over a small ∞-category $\mathscr{C}$ is obviously an ∞-topos.
(3) The most important (for an algebraic geometer, at least) class of examples for ∞-topoi is provided by ∞-*categories of derived stacks for some Grothendieck topology on derived commutative rings* in the sense of [Lur11, Definition 2.4.3], see for example [Lur11, Warning 2.4.5].

2.10. Some exercises.

(1) Again, try to prove some of the manifold statements which were left unproven in this section. Keep in mind that some of these results involve only minor combinatorial arguments, while other can prove themselves to be quite challenging – or worse. (Anyway, some of the most approachable exercises are explicitly presented in this list, see below.)
(2) Prove that the opposite quasi-category $\mathscr{C}^{\mathrm{op}}$ introduced in Remark 2.1.8.(4) indeed does deserve its name.
(3) Let X and Y be two sets, seen as discrete simplicial sets (i.e., $X_0 = X$ and X_n consists only of degenerate simplices for $n \geqslant 1$). Show that any map $f : X \to Y$ is a Kan fibration, and it is a trivial Kan fibration if and only if f is a bijection. Conclude that sets "do not need to be derived", hence ordinary categories are recovered in the homotopy theory of ∞-categories.
(4) (See [RV22, Exercise 1.1.v.].) Let Δ^1 be the *walking arrow category*, i.e., the (quasi-category associated to the) category consisting of two objects $[0]$ and $[1]$, and only one non-trivial arrow $f : [0] \to [1]$. Let $\mathbb{I}$ be the *walking isomorphism category*, i.e., the (quasi-category associated to the) category consisting of two objects $[0]$ and $[1]$, and two non-trivial arrows $f : [0] \to [1]$ and $f^{-1}\colon [1] \to [0]$ subject to the relations $f \circ f^{-1} = \mathrm{id}_{[1]}$ and $f^{-1} \circ f = \mathrm{id}_{[0]}$.
   a. Show that the quasi-category $\mathbb{I}$ admits precisely two non-degenerate 2-simplices.
   b. Show that the natural inclusion $\Delta^1 \subseteq \mathbb{I}$ is realized as a sequential composite of pushouts along inclusions of outer horns $\Lambda^n_0 \subseteq \Delta^n$ for $n \geqslant 2$.

c. Finally, show that an edge $f: x \to y$ in a quasi-category $\mathscr{C}$, corresponding to a quasi-functor $\Delta^1 \to \mathscr{C}$, is an equivalence precisely if it can be extended along the inclusion $\Delta^1 \subseteq \mathbb{I}$.

(5) Let $\mathscr{C}$ be a quasi-category, and let $\mathscr{C}_1'$ be a subclass of edges in $\mathscr{C}_1$ which is closed under composition, i.e., for any 2-simplex

$$\begin{tikzcd} & y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & {} & z \end{tikzcd}$$

if both f and g belong to $\mathscr{C}_1'$, then also h belongs to $\mathscr{C}_1'$. Let $\mathscr{C}'$ be the smallest full subsimplicial set of $\mathscr{C}$ spanned by $\mathscr{C}_1'$, i.e., an n-simplex $\sigma: \Delta^n \to \mathscr{C}$ lies in $\mathscr{C}'$ if and only if every edge $\Delta^1 \subseteq \Delta^n \to \mathscr{C}$ lies in $\mathscr{C}'$. Show that $\mathscr{C}'$ is a quasi-category.

(6) Show that the products and coproducts in a ∞-category $\mathscr{C}$ provide the homotopy 1-category $\mathrm{h}\mathscr{C}$ of categorical products and coproducts. Notice that we cannot extend this argument for general shapes of limits or colimits: the proof passes through the fact that

$$\mathrm{hFun}\left(\coprod_{i\in I}\{*\}, \mathscr{C}\right) \simeq \mathrm{Fun}\left(\coprod_{i\in I}\{*\}, \mathrm{h}\mathscr{C}\right),$$

which is in general false for arbitrary shapes. For example, is already false when the source is the walking arrow category Δ^1.

(7) Let $\mathscr{C}$ be a quasi-category, and let $\mathscr{C}_0'$ be a subclass of vertices in $\mathscr{C}_0$. Let $\mathscr{C}'$ be the smallest full subsimplicial set of $\mathscr{C}$ spanned by $\mathscr{C}_0'$, i.e., an n-simplex $\sigma: \Delta^n \to \mathscr{C}$ lies in $\mathscr{C}'$ if and only if every vertex $\Delta^0 \subseteq \Delta^n \to \mathscr{C}$ lies in $\mathscr{C}'$. Show that $\mathscr{C}'$ is a quasi-category and that the inclusion $\mathscr{C}' \subseteq \mathscr{C}$ is fully faithful.

(8) Let $\mathscr{C}$ be a quasi-category, and let $\mathscr{C} \to \widetilde{\mathscr{C}}$ be a fibrant replacement for the Quillen model structure on sSet. Prove that $\widetilde{\mathscr{C}}$ enjoys the following universal property: for any quasi-category $\mathscr{D}$, pre-composition with $\mathscr{C}$ induce a fully faithful quasi-functor

$$\mathrm{Fun}(\widetilde{\mathscr{C}}, \mathscr{D}) \longrightarrow \mathrm{Fun}(\mathscr{C}, \mathscr{D})$$

whose essential image is spanned by those quasi-functors $\mathscr{C} \to \mathscr{D}$ which send any edge of $\mathscr{C}$ to an equivalence in $\mathscr{D}$. Deduce that $\widetilde{\mathscr{C}} \simeq \mathscr{C}\left[\mathrm{Arr}(\mathscr{C})^{-1}\right]$.

(9) Let $\mathscr{C}$ be a quasi-category and let S be a small collection of edges in $\mathscr{C}_1$. Show that $\mathrm{h}\left(\mathscr{C}\left[S^{-1}\right]\right) \simeq \mathrm{h}\mathscr{C}\left[\overline{S}^{-1}\right]$, where $\overline{S}$ is the image of S under the canonical projection $\mathscr{C} \to \mathrm{h}\mathscr{C}$.

(10) (See also Remark 3.1.3) Let $\mathscr{C}$ a pointed ∞-category with zero-object 0. Suppose that for every object X in $\mathscr{C}$ the span

$$0 \longrightarrow X \longleftarrow 0$$

admits a limit ΩX and the cospan

$$0 \longleftarrow X \longrightarrow 0$$

admits a colimit ΣX. Construct explicitly ∞-functors $\Sigma\colon \mathscr{C} \to \mathscr{C}$ and $\Omega\colon \mathscr{C} \to \mathscr{C}$, and show that $\Sigma \dashv \Omega$ by explicitly constructing a both Cartesian and coCartesian fibration $\mathscr{M} \to \Delta^1$.

(11) Alternatively, prove the exercise before by showing that an ∞-functor $F\colon \mathscr{C} \to \mathscr{D}$ is a left adjoint *if and only if* for every object U in $\mathscr{D}$ there exists an object X in $\mathscr{C}$ and a natural transformation $\varepsilon_X : F(X) \to U$ such that for any other Y in $\mathscr{C}$ the composition

$$\mathrm{Map}_{\mathscr{D}}(Y, X) \xrightarrow{F} \mathrm{Map}_{\mathscr{C}}(F(Y), F(X)) \xrightarrow{\varepsilon_X \circ -} \mathrm{Map}_{\mathscr{C}}(F(Y), U)$$

is an equivalence.

3. Algebra in ∞-categories

Section 2 laid out all the foundations that we needed in order to work with ∞-categories. But we are interested in doing *algebra* in ∞-categories; hence, we need some class of ∞-categories in which it is natural to study algebraic phenomena – just like abelian categories provide the natural 1-categorical framework to perform homological algebra. In our case, this natural framework is provided by *stable ∞-categories*, and in particular the *∞-category of spectra*. For the following, it will be quite convenient to bear in mind this table of analogies, which will help shaping the intuition of the reader while developing the theory of *homotopical algebra*.

Ordinary algebra	**∞-categorical algebra**
Sets	Homotopy types
Abelian categories	Stable ∞-categories
Abelian groups	Spectra
Tensor product of abelian groups	Smash product of spectra
Associative algebras	Associative ($\mathbb{E}_1$-)ring spectra
Commutative algebras	Commutative ($\mathbb{E}_\infty$-)ring spectra

References for this section. From now on, the main reference for these notes will be [Lur17].

3.1. Stable ∞-categories.

Quite surprisingly, at this point we have already all the ∞-categorical notions we need in order to state in very simple terms what a stable ∞-category is.

Definition 3.1.1. Let $\mathscr{C}$ be a pointed ∞-category with zero object denoted by 0. A *triangle* in $\mathscr{C}$ is any commutative square in $\mathscr{C}$ of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z. \end{tikzcd}$$

We say that a triangle is a *fiber sequence* if it is a pullback square; in this case, X is said to be the *fiber* of the morphism $g \colon Y \to Z$, and is denoted by $\mathrm{fib}(g)$. Dually, we say that a triangle is said to be a *cofiber sequence* if it is a pushout square; in this case, Z is said to be the *cofiber* of the morphism $f \colon X \to Y$, and is denoted by $\mathrm{cofib}(f)$.

Notation 3.1.2. A triangle as in Definition 3.1.1 is the datum of a 2-simplex

$$\begin{tikzcd} & Y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & {} & Z \end{tikzcd}$$

together with a *null-homotopy* (i.e., another 2-simplex)

$$
\begin{tikzcd}
 & 0 \arrow[dr] & \\
X \arrow[ur] \arrow[rr, "h"'] \arrow[rr, phantom, shift left=3, "\Downarrow"] & & Z.
\end{tikzcd}
$$

In the following, we shall denote triangles by leaving implicit the data of the 2-simplices above, i.e., we shall simply write

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

Remark 3.1.3 ([Lur17, Remark 1.1.1.7.])**.** Both fibers and cofibers are well defined up to a contractible space of choices. Let $\mathrm{Fib}(\mathscr{C}) \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{C})$ be the inclusion of fiber sequences of $\mathscr{C}$ inside all commutative squares in $\mathscr{C}$. The morphism

$$\mathrm{ev}_{\{[0,1]\to[1,1]\}} \colon \mathrm{Fib}(\mathscr{C}) \longrightarrow \mathrm{Fun}(\Delta^1, \mathscr{C}),$$

which selects the "vertical arrow on the right" in a triangle, is a Kan fibration with either contractible or empty fibers. If moreover $\mathscr{C}$ admits all fiber sequences, then this inclusion is a trivial Kan fibration and therefore we have a section

$$\mathrm{Fun}(\Delta^1, \mathscr{C}) \xrightarrow{\simeq} \mathrm{Fib}(\mathscr{C}) \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{C})$$

which completes an arrow $f : X \to Y$ to a fiber sequence. By evaluating this commutative square at $[0,0]$, we obtain our desidered ∞-functor

$$\mathrm{fib} \colon \mathrm{Fun}(\Delta^1, \mathscr{C}) \longrightarrow \mathscr{C},$$

defined without assuming our ∞-category to admit *all* pullbacks. Similarly, one can define a cofiber ∞-functor. Notice that under these assumptions, we have adjunctions

$$\mathrm{cofib} \colon \mathrm{Fun}(\Delta^1, \mathscr{C}) \rightleftarrows \mathscr{C} \simeq \mathscr{C}_{0/} \simeq \mathrm{Fun}(\Delta^1, \mathscr{C}) \times_{\mathrm{ev}_0} \{0\} \colon \mathrm{R}$$

and

$$\mathrm{L} \colon \mathrm{Fun}(\Delta^1, \mathscr{C}) \times_{\mathrm{ev}_1} \{0\} \simeq \mathscr{C}_{0/} \simeq \mathscr{C} \rightleftarrows \mathrm{Fun}(\Delta^1, \mathscr{C}) \colon \mathrm{fib},$$

where the right adjoint in the first adjunction is the inclusion of an object X as the arrow $0 \to X$, while the left adjoint in the second adjunction is the inclusion of an object Y as the arrow $Y \to 0$. It follows that composing with adjoints in the "right direction" yields another adjunction

$$\Sigma \coloneqq \mathrm{cofib} \circ \mathrm{L} \colon \mathscr{C} \rightleftarrows \mathscr{C} \colon \mathrm{fib} \circ \mathrm{R} \eqqcolon \Omega. \tag{3.1.4}$$

Definition 3.1.5. Let $\mathscr{C}$ be a pointed ∞-category admitting fiber and cofiber sequences. The left adjoint $\Sigma \colon \mathscr{C} \to \mathscr{C}$ in the adjunction 3.1.4 is the *suspension ∞-functor*. The right ajdoint $\Omega \colon \mathscr{C} \to \mathscr{C}$ in the adjunction 3.1.4 is the *loop ∞-functor*.

Definition 3.1.6. A *stable ∞-category* is a pointed ∞-category $\mathscr{C}$, whose morphisms all admit both fibers and cofibers, such that any trangle in $\mathscr{C}$ is a fiber sequence if and only if it is also a cofiber sequence.

Notice that Definition 3.1.6 can be restated simply as follows: $\mathscr{C}$ is stable if it is pointed, admits all fiber and cofiber sequences, and the adjunction 3.1.4 is in fact an equivalence of ∞-categories. See also [Lur17, Corollary 1.4.2.27].

Remark 3.1.7. There are various analogies between stable ∞-categories and abelian categories.

(1) The condition that *fiber sequences are cofiber sequences* is a higher categorical version of the well known requirement for abelian categories: *the image of a morphism is canonically isomorphic to its coimage*.

(2) The axioms for stable ∞-categories and those for abelian categories are both obviously self-dual: $\mathscr{C}$ satisfies them if and only if $\mathscr{C}^{\mathrm{op}}$ does.

(3) Stability is a *property* of an ∞-category. Later, we shall see that this property endows stable ∞-categories of a natural enrichment over the stable ∞-category of spectra, just like abelian categories are defined in terms of categorical *properties*, yet are naturally endowed with the *structure* of an Ab-enriched cateagory.

Exercise 3.1.8 ([Lur17, Proposition 1.1.3.4])**.** Prove that a stable ∞-category is, in fact, both finitely complete and cocomplete using the criterion of Example 2.5.6.(6) as follows.

(1) First, prove that $\mathscr{C}$ admits binary coproducts: use [Lur09, Corollary 5.1.2.3], that guarantees that a diagram of arrows $p\colon K \to \mathrm{Fun}(\Delta^1, \mathscr{C})$ admits a colimit if and only if the induced diagrams $\mathrm{ev}_0 \circ p\colon K \to \mathscr{C}$ and $\mathrm{ev}_1 \circ p\colon K \to \mathscr{C}$ admit colimits. (*Hint: you do need cofibers!*)

(2) Next, prove that $\mathscr{C}$ admits pushouts. (*Hint: This is almost the same argument, word by word, used for proving that in additive categories pushouts are cokernels*.)

(3) By dual arguments, prove the same statements for binary products and pullbacks.

Actually, one can prove much more. First, prove that pullbacks and pushouts in ∞-categories enjoy all the known properties of ordinary 1-categorical pullbacks and pushouts: in particular, pullbacks and pushouts along equivalences are still equivalences, and the *pasting law for pullback and pushout diagrams* holds ([Lur09, Lemma 4.4.2.1]). In particular, deduce that in a stable ∞-category, composition of diagrams satisfy a stronger *two-out-of-three property*: consider a diagram $\Delta^1 \times \Delta^2 \to \mathscr{C}$, which we view as a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[rr, bend left] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\ U \arrow[r] \arrow[rr, bend right] & V \arrow[r] & W \end{tikzcd}$$

consisting of two small squares and a larger one obtained by pasting the other two. If two among these squares are pullbacks or pushouts, then so is the third. By toying around with pasted diagrams and these properties, deduce that in a stable ∞-category there is a canonical equivalence $X \times Y \simeq X \coprod Y$. In particular, stable ∞-categories are *additive*.

One of the most striking consequences of Remark 3.1.7.(3) is the following: the homotopy category of a stable ∞-category is *canonically* triangulated.

Construction 3.1.9. Let $\mathscr{C}$ be a stable ∞-category. Denote by $[n] := \Sigma^n$ the n-fold iteration of the suspension ∞-functor, and denote by $[-n] := \Omega^n$ the n-fold iteration of the loop ∞-functor: these define adjoint equivalences at the homotopy category level

$$[n]\colon \mathrm{h}\mathscr{C} \rightleftarrows \mathrm{h}\mathscr{C} : [-n].$$

Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

be a sequence of morphisms in $\mathrm{h}\mathscr{C}$. We shall say that the above is an *exact triangle* if there exists a diagram $\Delta^1 \times \Delta^2 \to \mathscr{C}$ of the form

$$\begin{tikzcd} X \arrow[r, "\widetilde{f}"] \arrow[d] & Y \arrow[r] \arrow[d, "\widetilde{g}"] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r, "\widetilde{h}"'] & W \end{tikzcd}$$

where both squares are pushouts, $\widetilde{f}$ and $\widetilde{g}$ represent f and g respectively, and h is given by the composition of the homotopy class of $\widetilde{h}$ with the canonical equivalence $W \simeq X[1]$.

Theorem 3.1.10 ([Lur17, Theorem 1.1.2.14])**.** *Let $\mathscr{C}$ be a stable ∞-category. Then the data of Construction 3.1.9 turn the homotopy category* $\mathrm{h}\mathscr{C}$ *into a triangulated category.*

Idea of proof. This is a tiresome, but straightforward, check that $\mathrm{h}\mathscr{C}$ satisfies the axioms of a triangulated category. The only part which, at least for people not used to homotopy theory, is not an easy exercise in strictifying and pasting diagrams from $\mathrm{h}\mathscr{C}$ to $\mathscr{C}$ is proving that $\mathrm{h}\mathscr{C}$ is additive. Exercise 3.1.8 guarantees that $\mathrm{h}\mathscr{C}$ admits all coproducts, so we are only left to prove that $\mathrm{h}\mathscr{C}$ is enriched over abelian groups. By the adjunction 3.1.4, for any couple of objects X and Y in $\mathscr{C}$ we have a natural equivalence of spaces

$$\mathrm{Map}_{\mathscr{C}}(\Sigma X, Y) \simeq \mathrm{Map}_{\mathscr{C}}(X, \Omega Y).$$

Using that Ω is a particular type of limit and that the Yoneda embedding よ$: \mathscr{C} \to \mathscr{S}$ preserves limits in $\mathscr{C}$, we obtain a chain of (natural) equivalences

$$\begin{aligned} \mathrm{Map}_{\mathscr{C}}(X, \Omega Y) &\simeq \mathrm{Map}_{\mathscr{C}}(X, 0 \times_Y 0) \\ &\simeq \mathrm{Map}_{\mathscr{C}}(X, 0) \times_{\mathrm{Map}_{\mathscr{C}}(X,Y)} \mathrm{Map}_{\mathscr{C}}(X, 0) \\ &\simeq \{*\} \times_{\mathrm{Map}_{\mathscr{C}}(X,Y)} \{*\} =: \Omega_* \mathrm{Map}_{\mathscr{C}}(X, Y), \end{aligned}$$

where $\Omega_* \colon \mathscr{S}_* \to \mathscr{S}_*$ is the loop ∞-functor of *pointed* spaces, and we are considering $\mathrm{Map}_{\mathscr{C}}(X, Y)$ to be pointed at the zero map. In particular, we have a chain of isomorphisms of homotopy groups

$$\begin{aligned} \pi_0 \mathrm{Map}_{\mathscr{C}}(\Sigma X, Y) &\cong \pi_0 \mathrm{Map}_{\mathscr{C}}(X, \Omega Y) \\ &\cong \pi_0 \Omega_* \mathrm{Map}_{\mathscr{C}}(X, Y) \cong \pi_1 \mathrm{Map}_{\mathscr{C}}(X, Y). \end{aligned}$$

Since this is a fundamental *group*, it follows that $\mathrm{Hom}_{\mathrm{h}\mathscr{C}}(X, Y) := \pi_0 \mathrm{Map}_{\mathscr{C}}(X, Y)$ is endowed with a natural group structure transferred by the one on $\pi_1 \mathrm{Map}_{\mathscr{C}}(X, Y)$. By choosing a double suspension $\Sigma^2 Z \simeq X$ functorially in X (we can do it, since Σ is an autoequivalence of $\mathscr{C}$), we obtain similarly another natural isomorphism

$$\pi_0 \mathrm{Map}_{\mathscr{C}}(X, Y) \cong \pi_2 \mathrm{Map}_{\mathscr{C}}(Z, Y)$$

hence this group structure is actually *abelian*, by the usual Eckmann-Hilton argument. □

Notation 3.1.11. Following the customs of homological algebra, given two objects X and Y in $\mathscr{C}$ we shall denote the group $\mathrm{Map}_{\mathscr{C}}(X[-n], Y)$ by $\mathrm{Ext}^n_{\mathrm{h}\mathscr{C}}(X, Y)$. If $n \leqslant 0$, this agrees with the $(-n)$-th homotopy group of $\mathrm{Map}_{\mathscr{C}}(X, Y)$. In particular, if $n = 0$ we shall simply denote $\mathrm{Ext}^0_{\mathrm{h}\mathscr{C}}(X, Y) \cong \pi_0 \mathrm{Map}_{\mathscr{C}}(X, Y)$ as $\mathrm{Hom}_{\mathrm{h}\mathscr{C}}(X, Y)$.

As in the case of presentable ∞-categories, we want to gather stable ∞-categories in some suitable ∞-category. In order to do so, we need to define what is the natural notion of ∞-functors between stable ∞-categories: in view of their definition, one could try the following.

Definition 3.1.12. An ∞-functor between stable ∞-categories $F \colon \mathscr{C} \to \mathscr{D}$ is *exact* if it preserves fibers and cofibers.

Yet, it is quite evident from Exercise 3.1.8 that, since all finite limits and colimits in stable ∞-categories are created by fibers and cofibers, we have the following.

Proposition 3.1.13 ([Lur17, Proposition 1.1.4.1])**.** *An ∞-functor between stable ∞-categories $F \colon \mathscr{C} \to \mathscr{D}$ is exact if and only if it is left exact, i.e., it preserves all finite limits, if and only if it is right exact, i.e., it preserves all finite colimits.*

Remark 3.1.14. In ordinary homological algebra, there is a distinction between left exact functors (preserving limits) and right exact functors (preserving colimits). In stable ∞-categories, this distinction is no more: this proves that we are, indeed, working in a *innerly derived* fashion.

We can now define the ∞-category Stab_∞ of stable ∞-categories and exact ∞-functors. This ∞-category admits all limits and filtered colimits, and the natural inclusion

$$\mathrm{Stab}_\infty \subseteq \widehat{\mathrm{Cat}}_\infty$$

preserves them.

Remark 3.1.15. Actually, Stab_∞ is a *presentable* ∞-category ([BGT13, Theorem 1.10]), hence it admits *all* colimits, even if they do not agree with colimits of their underlying ∞-categories. However, assuming to work with stable ∞-categories with a sufficient supply of infinite colimits, we can characterize cofibers in Stab_∞ as ∞-categorical enhancements of *Verdier quotients* of triangulated categories ([BGT13, Section 5.1]).

Theorem 3.1.10 already confirms the idea that it is natural to think of stable ∞-categories as "better behaved ∞-categorical enhancements of triangulated categories". Thus, we expect to be able to recover concepts from the theory of triangulated categories inside the theory of stable ∞-categories. In particular, we want to know what a *t-structure* should be.

Definition 3.1.16. A *t-structure* on a stable ∞-category $\mathscr{C}$ is a t-structure on its triangulated homotopy category $\mathrm{h}\mathscr{C}$, i.e., it is the datum of two full sub-categories $\mathrm{h}\mathscr{C}_{\geqslant 0}$ and $\mathrm{h}\mathscr{C}_{\leqslant 0}$ such that the following conditions hold.

(1) For any object X in $\mathrm{h}\mathscr{C}_{\geqslant 0}$ and any object Y in $\mathrm{h}\mathscr{C}_{\leqslant 0}$, one has $\mathrm{Hom}_{\mathrm{h}\mathscr{C}}(X, Y[-1]) \cong 0$.

(2) We have inclusions

$$\mathrm{h}\mathscr{C}_{\geqslant 1} := \mathrm{h}\mathscr{C}_{\geqslant 0}[1] \subseteq \mathrm{h}\mathscr{C}_{\geqslant 0}$$

and

$$\mathrm{h}\mathscr{C}_{\leqslant -1} := \mathrm{h}\mathscr{C}_{\leqslant 0}[-1] \subseteq \mathrm{h}\mathscr{C}_{\leqslant 0}.$$

(3) For any object X in $\mathrm{h}\mathscr{C}$, there exists a distinguished triangle of the form

$$X' \longrightarrow X \longrightarrow X''$$

where X' belongs to $\mathrm{h}\mathscr{C}_{\geqslant 0}$ and X'' belongs to $\mathrm{h}\mathscr{C}_{\leqslant -1}$.

We shall denote by $\mathscr{C}_{\geqslant 0}$ and by $\mathscr{C}_{\leqslant 0}$ the full sub-∞-categories of $\mathscr{C}$ spanned by $\mathrm{h}\mathscr{C}_{\geqslant 0}$ and $\mathrm{h}\mathscr{C}_{\leqslant 0}$, respectively. We shall say that X is *connective* if it belongs to $\mathscr{C}_{\geqslant 0}$, and that it is *coconnective* if it belongs to $\mathscr{C}_{\leqslant 0}$.

Exercise 3.1.17. Definition 3.1.16 looks somewhat underwhelming: there is no ∞-categorical content. One could guess that a more reasonable, ∞-categorical definition of a t-structure on a stable ∞-category $\mathscr{C}$ should be instead the datum of two full sub-∞-categories $\mathscr{C}_{\geqslant 0}$ and $\mathscr{C}_{\leqslant 0}$ such that the following conditions hold.

(1) For any object X in $\mathscr{C}_{\geqslant 0}$ and any object Y in $\mathscr{C}_{\leqslant 0}$, one has $\mathrm{Map}_{\mathscr{C}}(X, Y[-1]) \simeq 0$.

(2) We have inclusions

$$\mathscr{C}_{\geqslant 1} := \mathscr{C}_{\geqslant 0}[1] \subseteq \mathscr{C}_{\geqslant 0}$$

and

$$\mathscr{C}_{\leqslant -1} := \mathscr{C}_{\leqslant 0}[-1] \subseteq \mathscr{C}_{\leqslant 0}.$$

(3) For any object X in $\mathscr{C}$, there exists a fiber/cofiber sequence of the form

$$X' \longrightarrow X \longrightarrow X''$$

where X' belongs to $\mathscr{C}_{\geqslant 0}$ and X'' belongs to $\mathscr{C}_{\leqslant -1}$.

It is clear that these data imply a t-structure in the sense of Definition 3.1.16. Prove the converse.

Since the datum of a t-structure on a stable ∞-category is precisely the datum of a t-structure on its triangulated homotopy category, it is natural to expect that all the known results about t-structures on triangulated categories extend to the stable setting. And indeed, this is true.

Proposition 3.1.18 ([Lur17, Section 1.2.5])**.** *Let $\mathscr{C}$ be a stable ∞-category equipped with a t-structure.*

(1) The ∞-category $\mathscr{C}_{\leqslant 0}$ of coconnective objects of $\mathscr{C}$ is a localization, i.e., the inclusion $\mathscr{C}_{\leqslant 0} \subseteq \mathscr{C}$ is a right adjoint, whose left adjoint is the coconnective cover ∞-functor

$$\tau_{\leqslant 0} \colon \mathscr{C} \longrightarrow \mathscr{C}_{\leqslant 0}.$$

(2) *Dually, the inclusion of the ∞-category $\mathscr{C}_{\geqslant 0}$ of connective objects inside $\mathscr{C}$ is a left adjoint, whose right adjoint is the connective cover ∞-functor*

$$\tau_{\geqslant 0}\colon \mathscr{C} \longrightarrow \mathscr{C}_{\geqslant 0}.$$

(3) *Every fiber sequence*

$$X' \longrightarrow X \longrightarrow X''$$

where X' is connective and X'' is (-1)-coconnective is canonically equivalent to the fiber sequence

$$\tau_{\geqslant 0}X \longrightarrow X \longrightarrow \tau_{\leqslant -1}X.$$

(4) *The heart of $\mathscr{C}$*

$$\mathscr{C}^{\heartsuit} := \mathscr{C}_{\geqslant 0} \bigcap \mathscr{C}_{\leqslant 0}$$

is an abelian subcategory of $\mathscr{C}$. For any $n \in \mathbb{Z}$, the ∞-functor

$$\mathrm{H}_n\colon \mathscr{C} \xrightarrow{[n]} \mathscr{C} \xrightarrow{\tau_{\geqslant 0}\circ\tau_{\leqslant 0}} \mathscr{C}^{\heartsuit}$$

is the n-th homology ∞-functor.

3.2. **Models for algebraic stable ∞-categories.** In Section 3.1, we have seen how the homotopy category of stable ∞-categories is always triangulated. But we already know *plenty* of (algebraic) triangulated categories, so it is also natural to ask: given a triangulated category $\mathscr{T}$, does always exist a stable enhancement $\mathscr{T}^{\mathrm{enh}}$? Is it unique up to natural equivalence of ∞-categories? The answers to both questions is: *no, but almost yes*. First, let us introduce a bit of notation.

Definition 3.2.1. A *differential graded category* (from now on: *dg category*) over a base commutative ring $\Bbbk$ is a category $\mathscr{C}$ enriched over the category $\mathrm{C}_{\bullet}(\Bbbk)$ of chain complexes of $\Bbbk$-modules. In particular, $\mathscr{C}$ amounts to the following data.

(1) We have a class of *objects* of $\mathscr{C}$.
(2) For any couple of objects X and Y of $\mathscr{C}$ a *chain complex* of maps $\big(\mathrm{Map}_{\mathscr{C}}(X,Y)_{\bullet}, \mathrm{d}_{\bullet}\big)$.
(3) For any triple of objects X, Y and Z of $\mathscr{C}$ we have an associative composition law

$$\mathrm{Map}_{\mathscr{C}}(X,Y)_{\bullet} \otimes_{\Bbbk} \mathrm{Map}_{\mathscr{C}}(Y,Z)_{\bullet} \longrightarrow \mathrm{Map}_{\mathscr{C}}(X,Z)_{\bullet}$$

which amounts to a collection of $\Bbbk$-bilinear maps

$$\circ\colon \mathrm{Map}_{\mathscr{C}}(X,Y)_p \times \mathrm{Map}_{\mathscr{C}}(Y,Z)_q \longrightarrow \mathrm{Map}_{\mathscr{C}}(X,Z)_{p+q}$$

subject to the Leibniz rule $\mathrm{d}(g\circ f) = \mathrm{d}\,g\circ f + (-1)^p g\circ \mathrm{d}\,f$.
(4) For any object X of $\mathscr{C}$, we have a identity morphism $\mathrm{id}_X \in \mathrm{Map}_{\mathscr{C}}(X,X)_0$ which is a unit for the composition law.

A *dg functor* between dg categories $F\colon \mathscr{C} \to \mathscr{D}$ is a functor such that for any objects X and Y in $\mathscr{C}$ the map

$$\mathrm{Map}_{\mathscr{C}}(X,Y)_{\bullet} \longrightarrow \mathrm{Map}_{\mathscr{D}}(FX,FY)_{\bullet}$$

is a morphism of chain complexes. In this way, we can gather (small) dg categories and dg functors between them in a category that we denote by $\widehat{\mathrm{Cat}}_{\Bbbk}^{\mathrm{dg}}$.

Exercise 3.2.2. Prove that the Leibniz rule implies that the identity morphism is a cycle, i.e., $\mathrm{d}(\mathrm{id}) = 0$.

Remark 3.2.3. Every $\Bbbk$-linear category is a dg category over $\Bbbk$ in a stupid way, under the (strongly monoidal) functor from ordinary $\Bbbk$-modules to chain complexes which considers a $\Bbbk$-module as a chain complex concentrated in degree 0. On the converse, every dg category over $\Bbbk$ has an associated $\Bbbk$-linear category obtained by considering only the $\Bbbk$-module spanned by the 0-cycles in every mapping chain complex, i.e., for any objects X and Y in a dg category $\mathscr{C}$ we consider the Hom-space

$$\mathrm{Hom}_{\mathscr{C}}(X, Y) := \mathrm{Z}_0\left(\mathrm{Map}_{\mathscr{C}}(X, Y)\right) := \left\{f \in \mathrm{Map}_{\mathscr{C}}(X, Y)_0 \mid \mathrm{d}(f) = 0\right\}.$$

Indeed, the Leibniz rule requirement in Definition 3.2.1 implies that the composition restricts to the submodule of 0-cycles.

Definition 3.2.4. Let $\mathscr{C}$ be a dg category. The *homotopy category* of $\mathscr{C}$ is the $\Bbbk$-linear category $\mathrm{h}\mathscr{C}$ (or $\mathrm{H}_0(\mathscr{C})$) having the same class of objects as $\mathscr{C}$, and having as morphisms the $\Bbbk$-module

$$\mathrm{Hom}_{\mathrm{h}\mathscr{C}}(X, Y) := \mathrm{H}_0\left(\mathrm{Map}_{\mathscr{C}}(X, Y)_\bullet\right).$$

We say that a dg functor F is a *weak equivalence* if $\mathrm{h}F : \mathrm{h}\mathscr{C} \longrightarrow \mathrm{h}\mathscr{D}$ is an equivalence of ordinary categories and if for any couple of objects X and Y in $\mathscr{C}$ the natural map

$$\mathrm{Map}_{\mathscr{C}}(X, Y)_\bullet \longrightarrow \mathrm{Map}_{\mathscr{D}}(FX, FY)_\bullet$$

is a quasi-isomorphism.

Construction 3.2.5. Let $\mathscr{A}$ be a Grothendieck abelian category with enough projectives; this means that $\mathscr{A}$ is an abelian category admitting all colimits, such that filtered colimits are compatible with monomorphisms, and admitting a generator (i.e., an object R such that whenever $\mathrm{Hom}_{\mathscr{A}}(R,X) \cong 0$, then $X \cong 0$). Let us assume $\mathscr{A}$ to be $\Bbbk$-linear: every hom-set is a $\Bbbk$-module. Under these assumptions, the category $\mathrm{C}_\bullet(\mathscr{A}_{\mathrm{cf}})$ of (possibly unbounded) chain complexes of objects of $\mathscr{A}$ admits an injective model structure analogous to the one defined for chain complexes of $\Bbbk$-modules in Theorem 1.3.11. We produce a *differential graded* (*dg*) *enhancement* $\mathscr{A}^{\mathrm{dg}}$ of $\mathscr{A}$ as follows.

(1) The objects are the objects of $\mathrm{C}_\bullet(\mathscr{A})_{\mathrm{cf}}$, i.e., they are both fibrant and cofibrant chain complexes of objects of $\mathscr{A}$.
(2) For every couple of chain complexes $X_\bullet$ and $Y_\bullet$ over $\mathscr{A}$ as above we produce a mapping chain complex $\mathrm{Map}_{\mathscr{A}^{\mathrm{dg}}}(X_\bullet, Y_\bullet)_\bullet$ whose underlying graded $\Bbbk$-module is characterized by setting

$$\mathrm{Map}_{\mathscr{A}^{\mathrm{dg}}}(X_\bullet, Y_\bullet)_n := \prod_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{A}}\left(X_p, Y_{p+n}\right).$$

For f a map between $X_\bullet$ and $Y_\bullet$ of degree n, we define the differential $\mathrm{d}_n(f)$ as

$$\mathrm{d}_n(f) := \mathrm{d}^Y \circ f + (-1)^n f \circ \mathrm{d}^X .$$

Exercise 3.2.6. With the notations of Construction 3.2.5, for any couple of objects $X_\bullet$ and $Y_\bullet$ of $\mathscr{A}^{\mathrm{dg}}$ prove the following claims.

(1) We have a natural isomorphism
$$Z_0\left(\mathrm{Map}_{\mathscr{A}^{\mathrm{dg}}}(X_\bullet, Y_\bullet)\right) \cong \mathrm{Hom}_{C_\bullet(\mathscr{A})}(X_\bullet, Y_\bullet).$$
(2) We have a natural isomorphism
$$H_0\left(\mathrm{Map}_{\mathscr{A}^{\mathrm{dg}}}(X_\bullet, Y_\bullet)\right) \cong \mathrm{Hom}_{K(\mathscr{A})}(X_\bullet, Y_\bullet),$$
where $K\left(\mathscr{A}_{\mathrm{proj}}\right)$ is the naive homotopy category of $\mathscr{A}_{\mathrm{proj}}$. In particular, since $X_\bullet$ and $Y_\bullet$ were assumed to be bifibrant, it follows that
$$\mathrm{Hom}_{K(\mathscr{A})}(X_\bullet, Y_\bullet) \cong \mathrm{Hom}_{\mathscr{D}(\mathscr{A})}(X_\bullet, Y_\bullet).$$
In particular, the homotopy category of $\mathscr{A}^{\mathrm{dg}}$ recovers the usual derived category of $\mathscr{A}$.

Construction 3.2.7 ([Lur17, Construction 1.3.1.3])**.** Recall the Dold-Kan correspondence of Theorem 1.4.7. It is not strongly monoidal, yet thanks to the shuffle product of Alexander and Whitney one can regard this correspondence as *lax* monoidal, i.e., they still send monoid objects to monoid objects in a canonical way. In particular, consider $\mathscr{A}$ to be the category of abelian groups. Under the lax monoidal composition
$$C_\bullet(\mathscr{A}) \xrightarrow{\tau_{\geqslant 0}} C_{\geqslant 0}(\mathscr{A}) \xrightarrow{\mathrm{DK}} \mathrm{sAb} \xrightarrow{\mathrm{oblv}_{\mathrm{Ab}}} \mathrm{sSet},$$
obtained by cleverly truncating a chain complex in non-negative degrees, taking the associated simplicial abelian group, and then forgetting the abelian group structure, we can regard every differential graded category as a simplicial category, in particular as a ∞-category.

Theorem 3.2.8 ([Lur17, Proposition 1.3.1.9] and [Coh13])**.** *Let $\mathrm{L}^{\mathrm{H}}\left(\widehat{\mathrm{Cat}}^{\mathrm{dg}}_{\Bbbk}, \mathscr{W}\right)$ be the ∞-category of differential graded categories over $\Bbbk$ obtained by hammock-localizing the category of dg categories at the class of weak equivalences of Definition 3.2.4. Then the Dold-Kan construction of Construction 3.2.7 provides a ∞-functor*
$$\mathrm{L}^{\mathrm{H}}\left(\widehat{\mathrm{Cat}}^{\mathrm{dg}}_{\Bbbk}, \mathscr{W}\right) \longrightarrow \mathrm{Stab}_\infty.$$

Remark 3.2.9. Actually, by further localizing the category of differential graded $\Bbbk$-categories at a finer class of weak equivalences – i.e., *Morita equivalences* – then Theorem 3.2.8 can be refined as follows: the ∞-category of $\Bbbk$-linear dg categories up to Morita equivalence is equivalent to the ∞-category of idempotent-complete stable ∞-categories which are weakly enriched over the ∞-category of $\mathrm{H}\Bbbk$-modules in spectra. Yet, for now, almost nothing of this sentence makes any sense to the reader.

Theorem 3.2.10 ([Lur17, Theorem 1.3.3.2 and 1.3.5.21])**.** *Let $\mathscr{A}$ be a Grothendieck abelian category with enough projective objects, and let $\mathscr{D}(\mathscr{A})$ the stable derived ∞-category obtained from Construction 3.2.5 and Construction 3.2.7. Then $\mathscr{D}(\mathscr{A})$ admits a t-structure on $\mathscr{D}(A)$ such that $\mathscr{D}(\mathscr{A})_{\geqslant 0}$ is spanned by those chain complexes which have trivial homology in negative degrees and $\mathscr{D}(\mathscr{A})_{\leqslant 0}$ is spanned by those chain complexes which have trivial homology in positive degrees, whose heart is naturally equivalent to $\mathscr{A}$ itself. Moreover, if we consider the full sub-∞-category*
$$\mathscr{D}^{-}(\mathscr{A}) := \bigcup_{n \geqslant 0} \mathscr{D}(\mathscr{A})_{\geqslant -n} \subseteq \mathscr{D}(\mathscr{A}),$$

the restriction of the above t-structure defines a t-structure on $\mathscr{D}^-(\mathscr{A})$ which is universal in the following sense: let $\mathscr{C}$ be any stable ∞-category equipped with a left complete t-structure. Then there is an equivalence between the ordinary category of right exact functors $\mathrm{Fun}^{\mathrm{rex}}(\mathscr{A}, \mathrm{h}\mathscr{C}^{\heartsuit})$ *and the full sub-∞-category of* $\mathrm{Fun}(\mathscr{D}^-(\mathscr{A}), \mathscr{C})$ *spanned by those ∞-functors that send projective objects of $\mathscr{A}$ into the heart of the t-structure on $\mathscr{C}$.*

Warning 3.2.11. Theorem 3.2.10 tells us that if we consider a Grothendieck abelian category $\mathscr{A}$ with enough projectives (for example, the category of $\Bbbk$-modules over a ring) then there exists a dg category, hence a stable ∞-category, which "enhances" the usual derived category. In virtue of [CS18], we know that this is the *essentially unique way* in which we can lift the derived category to the ∞-categorical framework; it follows that some other important, "non-affine" triangulated categories (such as the unbounded derived category of an algebraic stack, or the category of perfect complexes over a Noetherian scheme) admit an essentially unique dg/stable enhancement as well. However, there are subtle issues in the non-affine case: stable derived ∞-categories satisfy descent for the étale, fppf and Nisnevich topologies, hence they can be glued and produce a well defined stable ∞-category of quasi-coherent sheaves for every derived scheme or derived stack. Such ∞-category is, in general, *not a derived ∞-category of an abelian category*, at all; however, if our derived stack is a classical Deligne-Mumford stack $\mathscr{X}$, it is true that the derived ∞-category is the stable enhancement of the usual derived category of $\mathscr{X}$. Many people have investigated the problem of the uniqueness of dg/stable enhancements of tringulated categories in the past twenty years or so, and many counterexamples, retaining different flavors, are known: see for example [Sch01, Section 2.1] for a topological example, [DS09] for an algebraic counterexample defined over $\mathbb{Z}$, and [RV19] for an algebraic counterexample defined over a field.

Remark 3.2.12. In some fields of research, for example symplectic geometry and theoretical physics, one can stumble upon the similar and more general (but equivalent at the level of homotopy theory) concept of A_∞-*categories*. An A_∞-category $\mathscr{C}$ is just like a dg category equipped with "higher composition laws", and can be defined as follows.

(1) Again, we have a class of objects of $\mathscr{C}$.
(2) For any couple of objects X and Y, we have a collection of graded $\Bbbk$-modules $\mathrm{Map}_{\mathscr{C}}(X, Y)_\bullet$.
(3) For any $n \geqslant 1$ and any collection $Y_1, \ldots, Y_n$ of objects of $\mathscr{C}$, using a *homological* grading convention, we have morphisms

$$\mathrm{m}_n \colon \mathrm{Map}_{\mathscr{C}}(X, Y_1)_\bullet \otimes_\Bbbk \mathrm{Map}_{\mathscr{C}}(Y_1, Y_2)_\bullet \otimes_\Bbbk \ldots \otimes_\Bbbk \mathrm{Map}_{\mathscr{C}}(Y_{n-1}, Y_n)_\bullet \longrightarrow \mathrm{Map}_{\mathscr{C}}(X, Y_n)_\bullet[2-n].$$

(4) With the previous notations, for any couple of objects X and Y and for any $n \geqslant 1$ the higher compositions are subject to relations

$$\sum_{n=r+s+t} (-1)^{r+st}\, \mathrm{m}_{r+t+1}\left(\mathrm{id}_X^{\otimes r} \circ \mathrm{m}_s \circ \mathrm{id}_Y^{\otimes t}\right) = 0.$$

In particular, for $n = 1$ we obtain a differential turning the graded $\Bbbk$-module $\mathrm{Map}_{\mathscr{C}}(X, Y)_\bullet$ into a chain complex, and for $n = 2$ we obtain some Leibniz rule for higher associativity, and so forth.

A_∞-functors are defined as expected. We have a natural functor from the category of dg categories to the category of A_∞-categories obtained by simply putting $m_n = 0$ for any $n > 1$. As proved in [COS19], this produces an equivalence of homotopy theories between differential graded categories and A_∞-categories, hence they are different ways to package the same homotopy-theoretic content. This does not mean that one between the two is a more convenient or more obsolete model!

3.3. **The stable ∞-category of spectra.** Despite our goal is to study (mainly) homological algebra in the context of ∞-categories, the most fundamental and *basic* – in a precise, algebraic sense – example of a stable ∞-category does not arise from homological algebra, rather from algebraic topology. The term *stable* comes itself from algebraic topology: *stable homotopy theory* is in fact the study of homotopical phenomena which, afters a reasonable number of steps, "stabilize". The most fundamental result in this regard is the following: let S^n be the n-dimensional sphere pointed at the unit vector e_1. The reduced suspension ΣS^n is homeomorphic to S^{n+1}, and we have a natural map $S^n \to \Omega\Sigma S^n$ given by the adjunction $\Sigma \dashv \Omega$. This produces, for any integer k, a map

$$\pi_{n+k}(S^n) \longrightarrow \pi_{n+k}(\Omega\Sigma S^n) \cong \pi_{n+k+1}(S^{n+1}). \tag{3.3.1}$$

Theorem 3.3.2 (Freudenthal Suspension Theorem)**.** *For $n \geqslant k+2$, the map 3.3.1 is an isomorphism of abelian groups.*

We can therefore start studying homotopy theory which stabilizes after applying the suspension functor for a certain number of times, for example the *stable homotopy groups of topological spaces*

$$\pi_n^s(X) := \operatorname*{colim}_{k\to\infty} \pi_{n+k}(\Sigma^k X).$$

Many interesting phenomena in topology, homotopy theory, even mathematical physics, albeit *unstable* in nature, can be interpreted in terms of *stable* homotopy theory: for example, the fact that S^n can admit a unital group structure induced by a division algebra structure on $\mathbb{R}^{n+1}$ is equivalent to the existence of a class of Hopf invariant 1 in the stable homotopy groups. Other applications can be the classification of manifolds up to homotopy or framed cobordism, or classification of exotic smooth structures on the spheres: see this interesting MathOverflow post [Sta] for more. In order to study stable homotopy theory, it can be useful to work in a framework in which the suspension becomes invertible, since in the stable range we do not distinguish anymore between a space X of its k-fold reduced suspension $\Sigma^k X$. This yields the homotopy category of *spectra*, which are a sort of "algebraization" of homotopy types, in a sense hinted at by the following example. In classical algebraic topology, we can define a topological space X to be an H-space if it is *almost* a topological group, i.e., if $e \in X$ is the identity element and $\mu\colon X \times X \to X$ its multiplication, we ask $\mu(e,-)$ and $\mu(-,e)$ to be only *homotopic* to the identity of X. A May's recognition principle (also called *May's Delooping Theorem*, see [May72]) states that for any $n \geqslant 1$, given any pointed topological space (Y, y) which is n-connected[1], the iterated loop space $\Omega^n Y$ is an n-fold H-space: i.e., it admits n compatible H-space structures. Moreover, any n-fold H-space arises, up to weak homotopy equivalence, in this way. One could

[1]This means that all its homotopy groups up to degree n are trivial.

ask then what happens if n goes to infinity: morally, an object X which is an ∞-fold H-space should admit all possible deloopings, i.e., it sholud be an ∞*-loop space*. A (connective) spectrum can be thought as the datum of an ∞-loop spaces together with a choice of all its possible deloopings. So, let us summarize how the stable homotopy category of spectra hSp behaves (or what properties we expect it to enjoy).

(1) It should contain a full subcategory spanned by ∞*-fold* H*-spaces*, i.e., by topological groups equipped with infinitely many group operations interacting nicely one with the other. We will see that this means that any spectrum is a *fully commutative homotopy coherent topological group*.

(2) The loop ∞-functor $\Omega\colon \mathrm{hSp} \to \mathrm{hSp}$ is an autoequivalence: we can never run out of deloopings to invert the looping of a spectrum. In particular, we should expect spectra to have also *negative* homotopy groups, in contrast to ordinary spaces. Indeed, if we have an ∞-loop space modeled as a Ω*-spectrum* $E_\bullet$ with $E_0 =: X$, then we have a chain of isomorphisms of homotopy groups

$$\pi_n(\Omega^\infty X) \cong \pi_{n+1}(\Omega^{\infty-1}(X)) \cong \ldots \cong \operatorname*{colim}_{k\to\infty} \pi_{n+k}(X).$$

In particular, homotopy groups of spectra do not compute classical homotopy groups: rather, they compute *stable homotopy groups*.

(3) As already mentioned in the introduction, spectra can also be thought of as objects corresponding to generalized cohomology theories in virtue of the Brown Representability Theorem. In particular, for any abelian group A we should be able to consider an associated spectrum HA, the *Eilenberg-Maclane spectrum of* A, which corresponds to the singular (co)homology with coefficients in A.

(4) The homotopy category of spectra is endowed with a symmetric monoidal structure given by the smash product of spectra,

An introduction to the relationship between spectra, stable homotopy theory, and generalized cohomology theories can be found in the excellent lecture notes [Sto22]. Here, our aim is somewhat different: rather than studying stable homotopy theory, we want to see how derived algebraic geometry and homological algebra live naturally in the stable homotopy category. Therefore, in this section we shall review the construction of the ∞-category of spectra (which is the ∞-categorical version of the abelian category of abelian groups) and determine some of its most fundamental properties.

Definition 3.3.3. Let $F : \mathscr{D} \to \mathscr{C}$ be an ∞-functor.

(1) If $\mathscr{D}$ admits pushouts, we say that F is *excisive* if it sends pushouts in $\mathscr{D}$ to pullbacks in $\mathscr{C}$.

(2) If $\mathscr{D}$ admits a final object $\mathbb{1}_{\mathscr{D}}$, we say that F is *reduced* if it sends $\mathbb{1}_{\mathscr{D}}$ to $\mathbb{1}_{\mathscr{C}}$.

If $\mathscr{D}$ admits a final object, we shall denote with $\mathrm{Fun}_*(\mathscr{D}, \mathscr{C})$ the full sub-∞-category of $\mathrm{Fun}(\mathscr{D}, \mathscr{C})$ spanned by reduced ∞-functors. If $\mathscr{D}$ admits pushouts, we shall denote with $\mathrm{Exc}(\mathscr{D}, \mathscr{C})$ the full sub-∞-category of $\mathrm{Fun}(\mathscr{D}, \mathscr{C})$ spanned by excisive ∞-functors. If $\mathscr{D}$ admits both final object and pushouts, we shall denote by $\mathrm{Exc}_*(\mathscr{D}, \mathscr{C})$ the full sub-∞-category of $\mathrm{Fun}(\mathscr{D}, \mathscr{C})$ spanned by both excisive and reduced ∞-functors.

Remark 3.3.4 ([Lur17, Remark 1.4.2.4])**.** If $\mathscr{D}$ is a small ∞-category which is pointed and admits all finite colimits, and $\mathscr{C}$ is presentable, then all the three ∞-categories above are presentable: they are actually *localizations* of the presentable ∞-category $\mathrm{Fun}(\mathscr{D}, \mathscr{C})$.

We are interested in the case when $\mathscr{D} := \mathcal{S}_*^{\mathrm{fin}}$, i.e., when the source is the ∞-category of *finite pointed spaces*. This is described as follows: let $\mathcal{S}^{\mathrm{fin}}$ be the smallest full sub-∞-category of spaces which contains the point $\{*\}$ and which is closed under finite colimits. Then

$$\mathcal{S}_*^{\mathrm{fin}} := \mathcal{S}_{*/}^{\mathrm{fin}}$$

is the ∞-category of pointed objects in $\mathcal{S}^{\mathrm{fin}}$. In particular, it is small, pointed, and admits all finite colimits.

Definition 3.3.5. Let $\mathscr{C}$ be an ∞-category. A *spectrum object* of $\mathscr{C}$ is an excisive and reduced ∞-functor $F\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathscr{C}$. We denote the ∞-category $\mathrm{Exc}_*\left(\mathcal{S}_*^{\mathrm{fin}}, \mathscr{C}\right)$ as $\mathrm{Sp}(\mathscr{C})$.

It is easy to see that if $\mathscr{C}$ admits all finite limits, then $\mathrm{Sp}(\mathscr{C})$ admits all finite limits as well, using the fact that limits in ∞-categories of ∞-functors are computed point-wise. Moreover, if $\mathscr{D}$ is pointed and admits all colimits, then $\mathrm{Exc}_*(\mathscr{D}, \mathscr{C})$ is pointed as well: under these assumptions, any reduced (and in particular, any excisive and reduced) ∞-functor F is a left Kan extension of the ∞-functor $F(\{*\})\colon \{*\} \to \mathscr{C}$ along the inclusion $\{*\} \hookrightarrow \mathscr{D}$.

Proposition 3.3.6 ([Lur17, Proposition 1.2.4.16])**.** *Let $\mathscr{C}$ be a finitely complete ∞-category. Then $\mathrm{Sp}(\mathscr{C})$ is stable, and therefore deserves the name of stabilization of $\mathscr{C}$.*

Remark 3.3.7. In ordinary category theory, if $\mathscr{C}$ admits products we can talk about *abelian group objects in $\mathscr{C}$*, i.e., objects G of $\mathscr{C}$ equipped with a multiplication $G \times G \to G$, an inverse map $G \to G$ and a point $\{*\} \to G$ which turn G into an internal abelian group in $\mathscr{C}$. The category of abelian group objects $\mathrm{Ab}(\mathscr{C})$, however, is not always abelian in general: notoriously, for example, $\mathrm{Ab}(\mathrm{Top})$ is not abelian. In the ∞-categorical setting, instead, admitting all limits is enough to guarantee that $\mathrm{Sp}(\mathscr{C})$ is stable.

For convenience, we spell out some important properties of the stabilization of a finitely complete ∞-category $\mathscr{C}$.

Proposition 3.3.8. *Let $\mathscr{C}$ be a finitely complete ∞-category.*

(1) The ∞-category $\mathscr{C}$ is stable if and only if the ∞-loop ∞-functor

$$\Omega^\infty\colon \mathrm{Sp}(\mathscr{C}) \longrightarrow \mathscr{C}$$

given by evaluating an excisive and reduced ∞-functor F on the 0-sphere S^0 is an equivalence of ∞-categories.

(2) Let $\mathscr{C}_*$ be the ∞-category of pointed objects in $\mathscr{C}$. Then postcomposition with the forgetful ∞-functor $\mathscr{C}_* \to \mathscr{C}$ yields an equivalence of ∞-categories $\mathrm{Sp}(\mathscr{C}_*) \simeq \mathrm{Sp}(\mathscr{C})$.

(3) Let $\mathscr{D}$ be a stable ∞-category, and let $\mathrm{Fun}^{\mathrm{lex}}(\mathscr{D}, \mathscr{C})$ be the full sub-∞-category of $\mathrm{Fun}(\mathscr{D}, \mathscr{C})$ spanned by left exact ∞-functors. Then composition with $\Omega^\infty\colon \mathrm{Sp}(\mathscr{C}) \to \mathscr{C}$ induces an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{lex}}(\mathscr{D}, \mathrm{Sp}(\mathscr{C})) \longrightarrow \mathrm{Fun}^{\mathrm{lex}}(\mathscr{D}, \mathscr{C}).$$

(4) Consider the tower of ∞-categories

$$\ldots \longrightarrow \mathscr{C} \xrightarrow{\Omega_{\mathscr{C}}} \mathscr{C} \xrightarrow{\Omega_{\mathscr{C}}} \mathscr{C}.$$

Then $\mathrm{Sp}(\mathscr{C})$ *identifies with the limit of such tower.*

Example 3.3.9. It may seem not immediately obvious what we just defined in Definition 3.3.5. We can explain the heuristics as follows: consider a (generalized and reduced) cohomology theory from the category CW_* of pointed CW complexes – for example, it can be the usual singular cohomology $\mathrm{H}^\bullet := \mathrm{H}^\bullet_{\mathrm{sing}}$.

(1) The *homotopy invariance* axiom implies that $\mathrm{H}^\bullet$ factors through hCW_*.

(2) The *additivity axiom* implies that $\mathrm{H}^\bullet$ turns arbitrary products into coproducts. Passing to the reduced theory, this translates in sending arbitrary wedge sums (the coproduct in pointed topological spaces) to products.

(3) The *exactness axiom*, saying that for any inclusion $\iota\colon A \hookrightarrow X$ one has a long exact sequence of cohomology groups relating the cohomology of A, the cohomology of X, and the relative cohomology of the pair (X,A), can be interpreted in terms of pointed topological spaces in the following way. Notice that one has an isomorphism of cohomology groups

$$\mathrm{H}^n(X,A) \cong \widetilde{\mathrm{H}}^n(\mathrm{cone}(\iota))$$

where $\mathrm{cone}(\iota)$ is the mapping cone on the inclusion ι. In particular, since the mapping cone produces the homotopy cofiber in the category of topological spaces (hence in the category of CW complexes), we are saying that

$$\mathbb{Z}[\mathrm{Sing}(\mathrm{cone}(\iota))] \longrightarrow \mathbb{Z}[\mathrm{Sing}(X)] \longrightarrow \mathbb{Z}[\mathrm{Sing}(A)]$$

is a homotopy pullback of simplicial abelian groups (hence, of simplicial sets). In particular, consider CX to be the cone on a topological space: then, CX is contractible (it retracts to its vertex) and the mapping cone of the inclusion $\iota\colon X \subseteq CX$ is homeomorphic to the reduced suspension ΣX. In particular, since a reduced cohomology theory is trivial when valued on the point, it follows that $\mathrm{H}^\bullet(X) \cong \mathrm{H}^{\bullet+1}(\Sigma X)$.

(4) Notice that the fact that $\mathrm{H}^n(X) \cong \mathrm{H}^{n+1}(\Sigma X)$ actually *forces* the abelian group structure on $\mathrm{H}^n(X)$. Indeed, ΣX is a cocommutative cogroup object in hCW_*, because for any other CW complex Y one has

$$\mathrm{Hom}_{\mathrm{hCW}_*}\big(\Sigma^2 X, Y\big) \cong \pi_2 \mathrm{Hom}_{\mathrm{Top}}(X, Y).$$

The additivity axiom applied to the $n+2$ degree of the cohomology theory, therefore, turns $\Sigma^2 X$ into an abelian group $\mathrm{H}^{n+2}(\Sigma^2 X)$. The exactness axiom (or better, the suspension isomorphism which is implied by the exactness axiom) then provides a natural abelian group structure on $\mathrm{H}^n(X) \cong \mathrm{H}^{n+2}(\Sigma^2 X)$ for every n.

Excisive and reduced ∞-functors from the ∞-category $\mathcal{S}_*^{\mathrm{fin}}$ with values in an ∞-category $\mathscr{C}$ with limits can therefore be interpreted as reduced cohomology theories on the ∞-category of finite pointed spaces with coefficients in $\mathscr{C}$: the homotopy invariance is naturally encoded in the ∞-categorical setting, while the exactness, additivity and suspensions axioms are encoded in the *excisive* condition. Finally, the ∞-categorical Brown Representability Theorem of [Lur17, Theorem 1.4.1.2] yields that a reduced cohomology theory functor from the ∞-category of finite pointed spaces with coefficients in $\mathscr{C}$ is always representable. Therefore, the ∞-category $\mathrm{Sp}(\mathscr{C})$ can be both interpreted as the ∞-category of reduced cohomology theories over finite pointed spaces with values in $\mathscr{C}$, or as the ∞-category of couples (X, n), where X is an object in $\mathscr{C}$ and n is an integer. The suspension ∞-functor $\Sigma_{\mathrm{Sp}(\mathscr{C})}$ simply sends (X, n) to $(X, n+1)$: this is obviously an autoequivalence.

The above discussion justifies the following fundamental definition.

Definition 3.3.10. The *stable ∞-category*, or the *∞-category of spectra*, is $\mathrm{Sp} := \mathrm{Sp}(\mathcal{S})$.

Some algebraic properties of the ∞-category of spectra can be spelled out only after a fast survey on the theory of ∞-operads and monoidal ∞-categories, that we postpone to Section 3.4. For the moment, we will settle for the following: some of these are immediate consequences of results mentioned in this section.

Definition 3.3.11. Consider the ∞-functor $\Omega^\infty\colon \mathrm{Sp} \to \mathcal{S}$. It is accessible, preserves all sifted colimits, and preserves all limits. In particular, it admits a left adjoint $\Sigma^\infty_+\colon \mathcal{S} \to \mathrm{Sp}$ that we call the *suspension spectrum ∞-functor*.

Proposition 3.3.12 ([Lur17, Proposition 1.3.4.6])**.** *Let $\mathrm{Sp}_{\geqslant 0}$ be the full sub-∞-category of* Sp *spanned under extensions and colimits by the essential image of Σ^∞_+. Dually, let $\mathrm{Sp}_{\leqslant -1}$ be the full sub-∞-category spanned by those spectra whose image under the ∞-loop ∞-functor is contractible. Then these data determine an accessible t-structure on* Sp *whose heart is equivalent to the abelian category of abelian groups.*

Remark 3.3.13. Proposition 3.3.12 deserves to be carefully analyzed.

(1) The fact that connective spectra come from the suspension spectrum of spaces is the ∞-categorical statement of the fact that ∞-loop spaces are equivalent to connective spectra: if X is a space, then $\Omega^\infty\Sigma^\infty_+ X$ is the *free ∞-group completion* X, and the homotopy groups of this ∞-loop space compute the stable homotopy groups of X,

(2) The coconnective spectra capture algebraic properties of stable homotopy groups that cannot come from the homotopy of ∞-loop spaces, since they are obviously concentrated in only non-negative degrees.

(3) For a base commutative ring $\Bbbk$, we have already defined a stable derived ∞-category $\mathscr{D}(\Bbbk)$. Moreover, Theorem 3.2.10 guarantees that the bounded stable derived ∞-category $\mathscr{D}(\Bbbk)$ is universal among stable ∞-categories equipped with a left complete t-structure whose heart is equivalent to the usual abelian category $\mathrm{Mod}_\Bbbk^\heartsuit$[2] of $\Bbbk$-modules. Hence, for any commutative ring $\Bbbk$, the exact functor

$$\mathrm{Mod}_\Bbbk^\heartsuit \xrightarrow{\mathrm{oblv}_\Bbbk} \mathrm{Ab} \hookrightarrow \mathrm{Sp}$$

produces an exact and conservative ∞-functor

$$\mathrm{H}\colon \mathscr{D}(\Bbbk) \longrightarrow \mathrm{Sp},$$

[2] This notation is surely peculiar, but it is motivated by the fact that we shall soon denote the full stable derived ∞-category $\mathscr{D}(\Bbbk)$ of $\Bbbk$-modules as $\mathrm{Mod}_\Bbbk$, see Notation 3.5.19.

that we call the *Eilenberg-MacLane spectrum ∞-functor*. This is a ∞-functor between presentable ∞-categories, hence by the Adjoint Functor Theorem 2.9.3 it admits both a left and right adjoint: in particular, its left adjoint will be realized as some sort of *relative tensor product* ∞-functor (Remark 3.5.13). However, in general it is not essentially surjective or fully faithful: for example, in Sp the mapping space $\mathrm{Map}_{\mathrm{Sp}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ has non-trivial higher homotopy groups, since there exist elements $\xi \in \pi_n \mathrm{Map}_{\mathrm{Sp}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \cong \mathrm{Ext}^{-n}_{\mathrm{Sp}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ corresponding to the (linear dual of) *stable cohomology operations*, i.e., natural transformations of cohomology theories

$$\mathrm{Sq}^n \colon \mathrm{H}\mathbb{Z}/p\mathbb{Z} \longrightarrow \mathrm{H}\mathbb{Z}/p\mathbb{Z}[n].$$

On the other hand, in $\mathscr{D}(\mathbb{Z}/p\mathbb{Z})$ the mapping space from $\mathbb{Z}/p\mathbb{Z}$ to $\mathbb{Z}/p\mathbb{Z}[n]$ is equivalent to $\mathbb{Z}/p\mathbb{Z}[n]$. It is however true that for $\Bbbk = \mathbb{Q}$ the Eilenberg-Maclane spectrum ∞-functor is fully faithful.

(4) When we will be able to talk about *monoidal* and *closed* ∞-categories, we will see that Sp is symmetric monoidal via the *smash product* and it is enriched over itself, i.e., there exists a full *spectrum* of maps between spectra (cfr. Remark 3.5.4). Considering the suspension spectrum $\Sigma^\infty_+ X$ over a space X and the spectrum E representing some sort of generalized (co)homology theory, we will see that $E \wedge \Sigma^\infty_+ X$ computes the homology of X with respect to E, while the mapping spectrum $\underline{\mathrm{Map}}_{\mathrm{Sp}}\big(\Sigma^\infty_+ X, E\big)$ computes the cohomology of X with respect to E. In particular, for $\Bbbk$ a ordinary commutative ring and for $E := \mathrm{H}\Bbbk$ then

$$\mathrm{C}_\bullet(X;\Bbbk) := \mathrm{H}\Bbbk \wedge \Sigma^\infty_+ X$$

computes the singular homology of X with coefficients in $\Bbbk$, and

$$\mathrm{C}^\bullet(X;\Bbbk) := \underline{\mathrm{Map}}_{\mathrm{Sp}}\big(\Sigma^\infty_+ X, \mathrm{H}\Bbbk\big)$$

computes the singular cohomology of X with coefficients in $\Bbbk$.

3.4. **Digression: monoidal ∞-categories and ∞-operads.** When doing classical algebraic geometry, the key ingredient of the theory is the concept of *commutative rings*, the fundamental objects which model – at least locally – schemes and algebraic spaces. Rings and commutative rings can be thought of as *monoids* and *commutative monoids* in the abelian category $\mathrm{Ab}^\otimes$ endowed with the symmetric monoidal structure given by the tensor product of abelian groups. In the ∞-categorical world, we have seen that the natural candidate for replacing the category of abelian groups is the stable ∞-category Sp of spectra, so we would like to define a symmetric monoidal structure generalizing the tensor product of abelian groups in order to consider commutative rings therein. In classical algebraic topology, the stable homotopy category hSp has indeed a monoidal structure given by the *smash product* so we would like to be able to put a ∞-categorical version of a smash product; yet we soon stumble upon some important obstacles messing with our goal: the coherencies and diagrams that we require to commute in order to define a monoidal structure are *way more* and *way more complicated* than one could imagine when first delving into homotopy theory, and things get worse if we want it to be commutative as well.

Example 3.4.1. In classical category theory and algebra, knowing the commutative behavior of the multiplication on any couple of elements in an associative group A is already enough to deduce the associativity for arbitrary n-uples: for example, in the case $n=3$, we know that

$$abc = a(bc) = a(cb) = (ac)b = (ca)b = c(ab) = c(ba) = (cb)a = (bc)a = b(ca) = b(ac).$$

Hence, if we denote by $\theta : A\times A \to A\times A$ the function that swaps factors in the Cartesian product and $\mathrm{m}^{(n)} : A^{\times n} \longrightarrow A$ the multiplication of n elements, knowing that there exists a 2-simplex (the identity) which fills the diagram

$$\begin{tikzcd}
A\times A \arrow[rr, "\theta"] \arrow[dr, bend right, "\mathrm{m}^{(2)}"'] & & A\times A \arrow[dl, bend left, "\mathrm{m}^{(2)}"] \\
& A &
\end{tikzcd}$$

is enough to deduce the existence of a 3-simplex (which is again the identity) which fills the diagram

$$\begin{tikzcd}
& A^{\times 3} \arrow[rr, "\theta\times\mathrm{id}"] \arrow[dddr, dotted] & & A^{\times 3} \arrow[dr, "\mathrm{id}\times\theta"] \arrow[dddl, dotted] & \\
A^{\times 3} \arrow[ur, "\mathrm{id}\times\theta"] \arrow[ddrr, "\mathrm{m}^{(3)}"'] & & & & A^{\times 3} \arrow[dl, "\theta\times\mathrm{id}"'] \arrow[ddll, "\mathrm{m}^{(3)}"] \\
& A^{\times 3} \arrow[ul, "\theta\times\mathrm{id}"] \arrow[dr, "\mathrm{m}^{(3)}"] & & A^{\times 3} \arrow[ll, "\mathrm{id}\times\theta"'] \arrow[dl, "\mathrm{m}^{(3)}"'] & \\
& & A. & &
\end{tikzcd}$$

This argument holds for *all* n-simplices. Yet, following the principle that in homotopical algebra *we need to remember all the homotopy coherencies*, passing to the ∞-categorical world things get messier. For instance, in the above example we have multiple choices of 2-simplices testifying to the commutativity of 2 elements (i.e., of homotopies between some way of multiplying three elements and some other), so it is not enough to specify one commutativity 2-simplex in order to deduce a commutativity 3-simplex, let alone to deduce all n-simplices for $n \geqslant 4$.

The above issue suggests another problem: in homotopy theory, an object can be *commutative, but not completely*. The reason why we do not see this phenomenon in ordinary categorical algebra can be motivated as follows: for the moment, we shall say that a monoid X in a symmetric monoidal ∞-category $\mathscr{C}$ is n-fold commutative if it is "commutative up to level n"[3], where we mean that X is associative if it is 1-fold commutative, and it is (completely) commutative if it is ∞-fold commutative. Then, objects in an n-category which are $(n+1)$-fold commutative are also $(n+k)$-fold commutative for any $k \geqslant 1$.

Example 3.4.2. A 2-commutative monoid in the category of sets is always fully commutative, as showed in the previous example. However, in the 2-category of ordinary (small) categories

[3] This may look esoteric – and for sure it is, in many regards – but actually this is way closer to the correct formalization than one could think at first glance: we shall soon see that what we really mean is that X is an *algebra for the* $\mathbb{E}_n$*-operad*, that we will describe in Construction 3.4.25.

we already taste the distinction, because we have 1-fold commutative monoids (monoidal categories), 2-fold commutative monoids (*braided* monoidal categories), and 3-fold commutative monoids (symmetric monoidal categories): there is a difference when the natural isomorphism $\theta^{-1}\circ\theta : X\times Y \to X\times Y$, and when such natural isomorphism is precisely the identity.

To sum it up: since one has to declare all the higher homotopies testifying to the degree of commutativity of a monoid, we have to settle with the fact that *commutativity becomes a piece of structure* on an associative monoid. This is quite an issue: in ordinary algebra, to specify a ring structure on an abelian group we simply say how the multiplication acts on two elements; a similar thing holds also when we want to specify a monoidal structure on a category. Now, we are instead asked of saying how the multiplication or the monoidal structure behaves with respect to *all* n-uples of objects. In order to face this problem, we shall work as follows.

(1) First, we encode all the coherencies that we want our operation to satisfy in a more compact and tractable combinatorial object, that we call an ∞-*operad* $\mathcal{O}$. Informally, this is a multicategory of *colors* (i.e., the amount of objects which are necessary to define the operation) together with all formal multilinear maps from an n-uple of colors to another one. (The fact that we need more than one color can be seen, for example, in the case of modules: we need to specify simultaneosuly both an algebra and the module over which it acts, see Construction 3.4.18.)
(2) If we have an ∞-category $\mathscr{C}$ with a natural candidate for a monoidal structure, we consider it as the underlying ∞-category of an ∞-operad $\mathscr{C}^{\otimes}$, which represents the "nebula" of coherencies that the monoidal structure of $\mathscr{C}$ must satisfy. In a very sketchy way, we could sum up this step by saying that we want to specify not only the tensor product on pairs of objects, but *arbitrary tensor products* on n-uples of objects of $\mathscr{C}$, for every $n \geqslant 2$.
(3) Finally, we define a reasonable notion of maps of ∞-operads and say that an algebra for the ∞-operad $\mathcal{O}$ in $\mathscr{C}$ is a map of ∞-operads $\mathcal{O}\to\mathscr{C}$ satisfying some properties.

The theory of operads is a, quite complicated indeed, tool that we introduce in order to be able to talk about symmetric monoidal ∞-categories and ∞-categories of algebras and modules; historically, they were introduced *precisely* to study objects equipped with operations and structures defined only up to homotopy, in order to study the theory of ∞-loop spaces. The reader is warned: this section is *highly* technical. However, we will see in Section 3.5 how in many cases of interest we can bypass this highly involved machinery and think of (some) algebras we want to study in the stable ∞-category as way more familiar algebraic objects defined in categories of chain complexes (see Theorems 3.5.18, 3.5.21 and 3.5.23).

Notation 3.4.3. For I a finite set, let $I_* := \{*\}\coprod I$ be the set I with an extra point added. Let Fin_* be the skeleton of the ordinary category of finite pointed sets $\mathrm{Set}_*^{\mathrm{fin}}$: i.e., objects are sets $\langle n\rangle := \{*,1,\dots,n\}$ for any $n\geqslant 0$ and maps $\langle n\rangle\to\langle m\rangle$ are point-preserving maps.

(1) We shall denote the set obtained by $\langle n\rangle$ *discarding* the base-point $\{*\}$ byf $\langle \mathring{n}\rangle$.
(2) We shall say that a map $f:\langle n\rangle\to\langle m\rangle$ is *inert* if for any element $i\in\langle \mathring{m}\rangle$ its preimage under f consists precisely of one element.
(3) We shall say that a map $f:\langle n\rangle\to\langle m\rangle$ is *active* if $f^{-1}(*)=\{*\}$.

(4) We shall denote by $\rho^i : \langle n\rangle \to \langle 1\rangle$ the unique inert map whose fiber over 1 is i.
(5) We shall denote by $\beta : \langle n\rangle \to \langle 1\rangle$ the unique active map.

Definition 3.4.4 ([Lur17, Definition 2.1.1.10 and Remark 2.1.1.14]). An ∞-*operad* is a ∞-functor $p : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ satisfying the following properties.

(1) Let $\mathcal{O}^{\otimes}_{\langle n\rangle}$ denote the fiber of $\langle n\rangle$ along p. Then for every object C in $\mathcal{O}^{\otimes}_{\langle n\rangle}$ and for every inert map $f : \langle n\rangle \to \langle m\rangle$ there exists a p-coCartesian morphism $\overline{f} : C \to C'$ lifting f.
(2) For any $f : \langle n\rangle \to \langle m\rangle$ and any C in $\mathcal{O}^{\otimes}_{\langle n\rangle}$ and C' in $\mathcal{O}^{\otimes}_{\langle m\rangle}$, let us denote by $\mathrm{Map}^f_{\mathcal{O}^{\otimes}}(C, C')$ be the union of the connected components of the space $\mathrm{Map}_{\mathcal{O}^{\otimes}}(C, C')$ which lie over f. Let $C' \to C'_i$ be a p-coCartesian lift of $\rho^i : \langle m\rangle \to \langle 1\rangle$ for all $1 \leqslant i \leqslant m$. Then composition induces a natural homotopy equivalence
$$\mathrm{Map}^f_{\mathcal{O}^{\otimes}}(C, C') \xrightarrow{\simeq} \prod_{1\leqslant i\leqslant m} \mathrm{Map}^{\rho^i \circ f}_{\mathcal{O}^{\otimes}}(C, C'_i).$$
(3) For each $n \geqslant 0$, the ∞-functors $\left\{\rho^i_! : \mathcal{O}^{\otimes}_{\langle n\rangle} \to \mathcal{O}^{\otimes}_{\langle 1\rangle}\right\}$ induce an equivalence of ∞-categories
$$\mathcal{O}^{\otimes}_{\langle n\rangle} \xrightarrow{\simeq} \mathcal{O}^{n}_{\langle 1\rangle}.$$

We shall often abuse notations when referring to an ∞-operad $\mathcal{O}^{\otimes}$ and avoid indicating the fiber map p.

Remark 3.4.5. What we just asked for in Definition 3.4.4? Let us review in a bit more detail those three conditions.

(1) Let us start from 3.4.4.(3). This guarantees that we have a "honest" ∞-category $\mathcal{O} := \mathcal{O}_{\langle 1\rangle}$, and the fibers $\mathcal{O}_{\langle n\rangle}$ are just n-fold products of $\mathcal{O}$: an object C in $\mathcal{O}_{\langle n\rangle}$ is just a collection of n objects $C_1 \oplus \ldots \oplus C_n$ lying in $\mathcal{O}$. In particular, for $n = 0$, we have $\mathcal{O}_{\langle 0\rangle} \simeq \mathcal{O}^0 \simeq \{*\}$: let us denote by $\mathbb{1}_{\mathcal{O}}$ the essentially unique object determined by this equivalence.
(2) For any inert map $f : \langle n\rangle \to \langle m\rangle$ in Fin_*, the existence of p-coCartesian lifts in Definition 3.4.4.(1) guarantees that we have a well defined ∞-functor
$$f_! : \mathcal{O}^n \longrightarrow \mathcal{O}^m.$$
This ∞-functor takes an object $C := (C_1, \ldots, C_n)$ to an object $C' := (C'_1, \ldots, C'_m)$ described informally as follows. If the preimage of $i \in \langle m\rangle$ under f is empty, then $C'_i := \mathbb{1}_{\mathcal{O}}$; if the preimage of i under f consists only of an element $f^{-1}(i)$, then $C'_i := C_{f^{-1}(i)}$.
(3) The homotopy equivalence in Definition 3.4.4.(2) guarantees that we do not need to know *all* maps $C_1 \oplus \ldots \oplus C_n \to C'_1 \oplus \ldots \oplus C'_m$, because we can recover them in an essentially unique way as a collection of m maps $C_1 \oplus \ldots \oplus C_n \to C_i$, with i ranging from 1 to m.

Notation 3.4.6. Let $p : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ be an ∞-operad. We shall denote by $\mathrm{Mul}_{\mathcal{O}}\left(\{C_i\}_{i\in\langle n\rangle^{\circ}}, D\right)$ the union of the connected components in $\mathrm{Map}_{\mathcal{O}^{\otimes}}\left(\{C_i\}_{i\in\langle n\rangle^{\circ}}, D\right)$ containing all the maps $C_1 \oplus \ldots \oplus C_n \to D$ lying over $\beta : \langle n\rangle \to \langle 1\rangle$.
This has to be thought of as a space of abstract multilinear maps which can be composed and satisfy some sort of axioms only up to coherent homotopy.

Example 3.4.7. We collect here some important examples and constructions of both ∞-operads and monoidal ∞-categories.

(1) The most elementary example of an ∞-operad is the *commutative ∞-operad* $\mathrm{Comm}^{\otimes} := \mathrm{Fin}_*$ itself: this provides an ∞-operad whose underlying ∞-category Comm consists of the single object $\mathfrak{a} := \langle 1\rangle$. For any possible object $\langle n\rangle$, we have a set of n-linear maps $\alpha\colon \langle n\rangle \to \langle 1\rangle$ which are completely determined by the choice of a subset $I \subseteq \langle \mathring{n}\rangle$ whose image under α is the singleton $\{1\}$. In particular, the map α is invariant under the action of any permutation $\sigma \in \Sigma_n$ which fixes I. Thinking of $\langle n\rangle$ as an abstract n-fold tensor product $\mathfrak{a}(1)\otimes\ldots\otimes\mathfrak{a}(n)$, the above discussion can be read as follows: for any $\sigma \in \Sigma_n$ such that $\sigma(I) = I$, the n-linear maps $\mathfrak{a}(1)\otimes\ldots\otimes\mathfrak{a}(n) \to \mathfrak{a}$ and $\mathfrak{a}(\sigma(1))\otimes\ldots\otimes\mathfrak{a}(\sigma(n)) \to \mathfrak{a}$ are the same. In particular, if $I = \{1,\ldots,n\}$ is the whole set $\langle \mathring{n}\rangle$, then we have a unique n-linear map $\mathfrak{a}(1)\otimes\ldots\otimes\mathfrak{a}(n) \to \mathfrak{a}$ (corresponding to $\beta\colon \langle n\rangle \to \langle 1\rangle$) which is invariant under any permutation of the factors in the domain. This justifies the adjective *commutative*: this ∞-operad indeed encodes a *fully commutative* multiplication.

(2) In classical category theory, one defines *colored operads* as a collection of *colors* (or *objects*), together with a set of multilinear maps

$$\sigma \in \mathrm{Mul}\big(\{X_i\}_{i=1,\ldots,n}\,, Y\big)$$

for any $(n+1)$-uple of colors X_i and Y, which can be composed in an associative way and such that by discarding n-linear maps for $n \geqslant 2$ one gets a honest category. These, in particular, are all ∞-operads.

(3) One notable example of a classical operad is the *associative operad* $\mathrm{Assoc}^{\otimes}$, which is the operad with same objects as Fin_*, but where morphisms $\langle n\rangle \to \langle m\rangle$ correspond to pairs consisting of a honest map α between finite pointed sets, *together with a linear ordering* over the fiber $\alpha^{-1}(j)$ for any $j \in \langle m\rangle$. To be clearer, let $\mathfrak{a}$ denote the element $\langle 1\rangle$ (which corresponds to the unique object of the underlying ∞-category $\Delta^0 \simeq \mathrm{Assoc}$ of such ∞-operad): then, for any n we ask $\mathrm{Mul}_{\mathrm{Assoc}}\big(\{\mathfrak{a}\}_{1\leqslant i\leqslant n}\,, \mathfrak{a}\big)$ to be the set of linear orderings of $\{1,\ldots,n\}$: for any integer n we have $n!$ possible linear orderings, hence $n!$ possible n-linear operations $\mathfrak{a}^{\otimes n} \to \mathfrak{a}$, corresponding to all the possible ways we can permute the factors in the tensor product $\mathfrak{a}^{\otimes n}$ (*a priori*, not equivalent one to the other).

(4) Another classical example is the *Lie operad* Lie: in general, since the axioms for the Lie bracket involve an additive structure, one has to work with the $\mathbb{L}_\infty$-operad, but if one is interested in ordinary (or homotopy-theoretic versions of the ordinary) Lie algebras in $\mathbb{k}$-modules we can present the operad Lie as the *$\mathbb{k}$-linear operad* on one object $\mathbb{1}$ and generated by a single 2-ary operation

$$[-,-] \in \mathrm{Mul}_{\mathrm{Lie}}(\{\mathbb{1},\mathbb{1}\}\,,\mathbb{1})$$

satisfying relations

$$[-,[-,-]]+[-,[-,-]]\tau+[-,[-,-]]\tau^2 = 0 \quad \text{and} \quad [-,-]+[-,-]\theta = 0,$$

where τ is a 3-cycle in Σ_3 and θ is a 2-cycle of Σ_2.

Notation 3.4.8. Given an ∞-operad $\mathcal{O}^{\otimes} \to \mathrm{Fin}_*$, we shall say that a map f in $\mathcal{O}^{\otimes}$ is inert if it is a p-coCartesian lift of an inert map $p(f)$ of Fin_*. We shall say that a map f is active if it is a lift of an active map $p(f)$.

Definition 3.4.9.

(1) Let $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ two ∞-operads. A *map of ∞-operads* is an ∞-functor $\mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ which commutes with the fiber maps over Fin_* and which preserves inert morphisms.

(2) Let $p\colon \mathscr{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-categories, where $\mathcal{O}^{\otimes}$ is an ∞-operad. We will say that p is a *coCartesian fibration of ∞-operads* if the composition $\mathscr{C}^{\otimes} \to \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ turns $\mathscr{C}^{\otimes}$ into an ∞-operad. In this case, we shall say that $\mathscr{C} := \mathscr{C}^{\otimes}_{\langle 1\rangle}$ is a *$\mathcal{O}$-monoidal ∞-category*.

(3) Let $\mathscr{C}^{\otimes}$ be an $\mathcal{O}$-monoidal ∞-category, and let $\mathcal{O}' \to \mathcal{O}$ be any map of ∞-operads. We denote by $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathscr{C})$ the sub-∞-category of $\mathrm{Fun}(\mathcal{O}', \mathscr{C})$ spanned by the maps of ∞-operads which commute with the projections to $\mathcal{O}$, and we call it the *∞-category of $\mathcal{O}$-algebra objects in $\mathscr{C}$*.

Remark 3.4.10. Again, it is useful to look a bit more carefully at Definition 3.4.9.(1). First, [Lur17, Proposition 2.1.2.12] tells us that the condition of $\mathscr{C}^{\otimes} \to \mathcal{O}^{\otimes}$ being a coCartesian fibration of ∞-operads is equivalent to asking that for any map $f \in \mathrm{Mul}_{\mathcal{O}^{\otimes}}\big(\{X_i\}_{1\leqslant i\leqslant n}, Y\big)$ we have a choice of coCartesian lifts which provide an ∞-functor $\mathscr{C}_{\{X_i\}_{1\leqslant i\leqslant n}} \to \mathscr{C}_Y$. If moreover such f is an inert map $\{X_i\}_{1\leqslant i\leqslant n} \to X_i$ in $\mathcal{O}^{\otimes}$, then the induced ∞-functor

$$f_!\colon \mathscr{C}_{\{X_i\}_{1\leqslant i\leqslant n}} \xrightarrow{\simeq} \prod_{1\leqslant i\leqslant n} \mathscr{C}_{X_i}$$

is an equivalence of ∞-categories. So, suppose we have a $\mathcal{O}$-monoidal ∞-category $\mathscr{C}^{\otimes}$. An $\mathcal{O}'$-algebra object is a map $\mathcal{O}'^{\otimes} \to \mathscr{C}^{\otimes}$ which commutes with the projections $p\colon \mathscr{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and $\mathcal{O}' \to \mathcal{O}$ which selects an object C_X in $\mathscr{C}$ for any object X in $\mathcal{O}'$, and such that for any n-linear map $f \in \mathrm{Mul}_{\mathcal{O}'^{\otimes}}\big(\{X_i\}_{i\in\langle n\rangle}, Y\big)$ we have an ∞-functor that we will denote by

$$\otimes_f\colon \prod_{1\leqslant i\leqslant n} C_{X_i} \longrightarrow C_Y.$$

The kind of conditions that these maps have to satisfy (associativity, commutativity...) is encoded in all the possible, different choices of coCartesian lifts of n-linear maps $\{X_i\} \to Y$. We shall see some explicit computations in Example 3.4.13 and in Example 3.4.26.

We are ready to state the definition of a *symmetric monoidal ∞-category*, which is a particular case of Definition 3.4.9.(2).

Definition 3.4.11.

(1) A *monoidal ∞-category* $\mathscr{C}^{\otimes}$ is an Assoc-monoidal ∞-category.
(2) A *symmetric monoidal ∞-category* $\mathscr{C}^{\otimes}$ is a Comm-monoidal ∞-category.

In both cases, we shall denote by $\mathscr{C}$ the underlying ∞-category of $\mathscr{C}^{\otimes}$, i.e., the fiber of the coCartesian fibration $\mathscr{C}^{\otimes} \to \mathrm{Comm}^{\otimes}$ over $\langle 1\rangle$.

(3) Let $p\colon \mathscr{C}^{\otimes} \to \mathrm{Fin}_*$ and $\mathscr{D}^{\otimes} \to \mathrm{Fin}_*$ be two symmetric monoidal ∞-categories. A *lax monoidal ∞-functor* is a map of ∞-operads $F\colon \mathscr{C}^{\otimes} \longrightarrow \mathscr{D}^{\otimes}$.

(4) Let $p\colon \mathscr{C}^{\otimes} \to \mathrm{Fin}_*$ and $\mathscr{D}^{\otimes} \to \mathrm{Fin}_*$ be two symmetric monoidal ∞-categories. A *strongly monoidal ∞-functor* is a lax monoidal ∞-functor which carries p-coCartesian morphisms in $\mathscr{C}^{\otimes}$ to q-coCartesian morphisms in $\mathscr{D}^{\otimes}$.

Remark 3.4.12 (Heuristics on Definition 3.4.11)**.** Let us look at Definition 3.4.11.(2). In virtue of the discussion provided in Remark 3.4.10, it is clear that since $\rho^i\colon \langle n\rangle \to \langle 1\rangle$ is inert for every n and every i, we have simply imposed Condition 3.4.4.(3) in Definition 3.4.4 to be satisfied for the ∞-operad $\mathscr{C}^{\otimes}$, i.e., we have asked for an equivalence

$$\langle \rho^i\rangle_{1\leqslant i\leqslant n}\colon \mathscr{C}^{\otimes}_{\langle n\rangle} \xrightarrow{\simeq} \mathscr{C}^{\times n}.$$

Moreover, the coCartesianity condition on the fibration guarantees that for all the active morphisms $\langle n\rangle \to \langle 1\rangle$ collapsing all elements except the base point to 1, we have an *n-fold tensor product ∞-functor*

$$\otimes\colon \mathscr{C}^{\times n} \longrightarrow \mathscr{C}$$

which is moreover invariant under the autoequivalence

$$\mathscr{C}^{\times n} \simeq \mathscr{C}^{\times n}$$

obtained by the coCartesian lift of any permutation of the set $\{1,\dots,n\}$, since projecting the set $\{1,\dots,n\}$ to the singleton $\{1\}$ is invariant under any permutation of the set $\{1,\dots,n\}$. In particular, we are imposing a *homotopy-coherent commutative monoidal structure on $\mathscr{C}$*.
On the other hand, if we ask only for a *monoidal* ∞-category $\mathscr{C}^{\otimes}$ according to Definition 3.4.11.(1), we have a coCartesian fibration $\mathscr{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}$. Notice that in $\mathrm{Assoc}^{\otimes}$ the inert morphisms $\langle n\rangle \to \langle 1\rangle$ are still the same inert morphisms as in $\mathrm{Comm}^{\otimes}$, but the collection of *all* n-linear morphisms is *larger*: for any morphism $\alpha\colon \langle n\rangle \to \langle 1\rangle$ in $\mathrm{Comm}^{\otimes}$, we have $n!$ maps $(\alpha,\preccurlyeq)\colon \langle n\rangle \to \langle 1\rangle$ in $\mathrm{Assoc}^{\otimes}$ lying over α, corresponding to all possible linear orderings on $\{1,\dots,n\}$. In particular, a lift of any of those $n!$ different n-linear maps $(\alpha,\preccurlyeq)$ yields a well-defined tensor product ∞-functor

$$(\alpha,\preccurlyeq)_!\colon \mathscr{C}^{\times n} \longrightarrow \mathscr{C}.$$

However, *a priori* this monoidal structure is not invariant under shuffles of factors anymore, because shuffling does not respect linear orderings; this means that we are only imposing a *homotopy-coherent associative monoidal structure on $\mathscr{C}$*. Notice that if we start with a symmetric monoidal ∞-category $\mathscr{C}^{\otimes} \to \mathrm{Comm}^{\otimes}$, the fibration $\mathrm{Assoc}^{\otimes} \to \mathrm{Comm}^{\otimes}$ allows us to forget $\mathscr{C}^{\otimes}$ to a merely monoidal ∞-category $\mathscr{C}^{\otimes} \times_{\mathrm{Comm}^{\otimes}} \mathrm{Assoc}^{\otimes}$, whose underlying ∞-category is the same as the one of $\mathscr{C}^{\otimes}$ (because over $\langle 1\rangle$, the fibration $\mathrm{Assoc}^{\otimes} \to \mathrm{Comm}^{\otimes}$ is an equivalence of ∞-categories).
Finally, let us focus on Definitions 3.4.11.(3) and 3.4.11.(4). A lax monoidal ∞-functor is a quite fancy way to encode the idea of an ∞-functor which, even if does not preserve the monoidal structure, still is sufficiently compatible with it, in the following sense. For every objects C and C' of $\mathscr{C} := \mathscr{C}^{\otimes}_{\langle 1\rangle}$, and denoting by $\mathbb{1}_{\mathscr{C}}$ and $\mathbb{1}_{\mathscr{D}}$ the unit for the monoidal structures on $\mathscr{C}$ and $\mathscr{D}$ respectively, one has maps

$$\mathbb{1}_{\mathscr{D}} \longrightarrow F(\mathbb{1}_{\mathscr{C}}) \quad \text{and} \quad F(C)\otimes_{\mathscr{D}} F(C') \longrightarrow F(C\otimes_{\mathscr{C}} C')$$

which satisfy some coherency conditions. In particular, lax monoidal ∞-functors send algebras for *any* operad $\mathcal{O}$ in $\mathcal{C}$ to algebras for the same operad in $\mathcal{D}$. A strongly monoidal ∞-functor is a lax monoidal ∞-functor for which the above maps are actually equivalences. In particular, strongly monoidal ∞-functors preserve not only algebras for any operad, but also *coalgebras* for any *cooperad*.

Example 3.4.13. Let us study more carefully, for some ∞-operads $\mathcal{O}$ of special interest, what a $\mathcal{O}$-algebra in a (symmetric) monoidal ∞-category $\mathcal{C}$ is.

(1) Given a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, the ∞-category $\mathrm{Alg}_{\mathrm{Comm}}(\mathcal{C})$ of Comm-algebras is the homotopy coherent way to define the ∞-category of *(unitary) commutative algebras in* $\mathcal{C}$, hence we shall denote it by $\mathrm{CAlg}(\mathcal{C})$. Indeed, a Comm-algebra is just a section of the fibration $\mathcal{C}^{\otimes} \to \mathrm{Comm}^{\otimes}$, which corresponds to the choice of an object A of $\mathcal{C}$ (corresponding to the image of $\langle 1\rangle$), with a pointing by the unit $\mathbb{1}_{\mathcal{C}} \to A$ (corresponding to the unique map $\langle 0\rangle \to \langle 1\rangle$) and a collection of multiplication maps

$$A^{\otimes n} \longrightarrow A$$

corresponding to all possible n-linear maps $\langle n\rangle \to \langle 1\rangle$. The fact that they are all homotopy-coherently associative and commutative is a consequence of an analogous discussion to the one of Remark 3.4.12: therefore, we shall call Comm-algebras in $\mathcal{C}^{\otimes}$ simply as *commutative algebras of* $\mathcal{C}$.

(2) Let us now consider the associative ∞-operad $\mathrm{Assoc}^{\otimes}$ of Example 3.4.7.(3): let us recall that this is the operad on one object $\mathfrak{a}$ such that for any n the set of n-linear maps $\mathrm{Mul}_{\mathrm{Assoc}}\big(\{\mathfrak{a}\}_{1\leqslant i\leqslant n}, \mathfrak{a}\big)$ is the set of *linear orderings* of $\{1,\ldots,n\}$. In particular, an Assoc-algebra in a monoidal ∞-category $\mathcal{C}^{\otimes}$ is a section $\mathrm{Assoc}^{\otimes} \to \mathcal{C}^{\otimes}$ of the coCartesian fibration $\mathcal{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}$. This boils down to the datum of an object A of $\mathcal{C}$ corresponding to $\mathfrak{a}$ together with a set of $n!$ possible n-linear operations $A^{\otimes n} \to A$ for any n. If we denote the j-th factor in $A^{\otimes n}$ as $A(j)$, so to write $A^{\otimes n}$ as $A(1)\otimes\ldots\otimes A(n)$, then it is easy to see that for any linear ordering of $\{1,\ldots,n\}$ corresponding to a permutation in $\theta \in \Sigma_n$ there exists a *unique* multiplication map

$$A(\theta(1))\otimes\ldots\otimes A(\theta(n)) \longrightarrow A.$$

This means that, as long as we keep the indices fixed, there exists a unique map $A^{\otimes n} \to A$ (i.e., it does not matter *where we put the brackets in multiplication*), but whenever we swap some of the factors we have another, *a priori* inequivalent, multiplication. We can see that this operad models *precisely* the associative algebras, hence we shall denote $\mathrm{Alg}_{\mathrm{Assoc}}(\mathcal{C})$ simply as $\mathrm{Alg}(\mathcal{C})$ and we will shall refer to Assoc-algebras in $\mathcal{C}$ as *associative algebras of* $\mathcal{C}$.

(3) In ordinary category theory, any category $\mathcal{C}$ admitting all products or coproducts admits a *Cartesian* or *coCartesian* symmetric monoidal structure, respectively, where the monoidal structure is given by taking products and coproducts. This holds in the ∞-categorical setting as well: if $\mathcal{C}$ is a ∞-category admitting all limits or colimits, then there is an essentially unique way to promote $\mathcal{C}$ to a Cartesian ($\mathcal{C}^{\times}$) or coCartesian ($\mathcal{C}^{\amalg}$) symmetric monoidal ∞-category, where the monoidal structure is given by products or

coproducts respectively. In this case, we have a natural equivalence of ∞-categories

$$\mathscr{C} \simeq \mathrm{cCcAlg}(\mathscr{C}^{\times}) \quad \text{and} \quad \mathscr{C} \simeq \mathrm{CAlg}(\mathscr{C}^{\amalg}),$$

where cCcAlg denotes the ∞-category of cocommutative coalgebras of $\mathscr{C}$[4]. See [Lur17, Proposition 2.4.1.5].

(4) In particular, $\widehat{\mathrm{Cat}}_{\infty}$ and $\mathcal{S}$ admit a Cartesian symmetric monoidal structure. At this point, one could ask whether the notion of $\mathscr{O}$*-algebras in* $\widehat{\mathrm{Cat}}_{\infty}$ and the notion of $\mathscr{O}$*-monoidal* ∞*-categories* coincide. Let us remark that a $\mathscr{O}$-algebra in $\widehat{\mathrm{Cat}}_{\infty}$ is a map of ∞-operads $\mathscr{O}^{\otimes} \to \widehat{\mathrm{Cat}}_{\infty}^{\times}$, which in particular has an "underlying ∞-functor" $\mathscr{O} \to \widehat{\mathrm{Cat}}_{\infty}$. By the ∞-categorical Grothendieck construction of Theorem 2.7.13, this is equivalent to a coCartesian fibration of ∞-categories $\mathscr{C} \to \mathscr{O}$. Such coCartesian fibration can be promoted to a coCartesian fibration of ∞-operads *precisely* if the corresponding ∞-functor $\mathscr{O} \to \widehat{\mathrm{Cat}}_{\infty}$ can be promoted to a map of ∞-operads: see for example [Lur17, Example 2.4.2.4]. Hence, the two notions *coincide*.

(5) Notice that the ∞-category of $\mathscr{O}'$-algebras in a $\mathscr{O}$-monoidal ∞-category $\mathscr{C}$ is, in particular, a ∞-category of ∞-functors. By evaluating an ∞-functor on *any* object X in $\mathscr{O}'$ one has a forgetful ∞-functor $\mathrm{Alg}_{\mathscr{O}'}(\mathscr{C}) \to \mathscr{C}$ which can be promoted to a map of ∞-operads over $\mathscr{O}$. This is simply a fancy way to say that if $\mathscr{C}$ is $\mathscr{O}$-monoidal, then the ∞-category of $\mathscr{O}'$-algebras for any ∞-operad $\mathscr{O}'$ is again $\mathscr{O}$-monoidal via the underlying monoidal structure of $\mathscr{C}$, and the forgetful ∞-functor is strongly monoidal. For an *even fancier* way to put it, see [Lur17, Example 3.2.4.4].

We want to spell out some general properties of ∞-categories of algebras in a monoidal ∞-category $\mathscr{C}$. In the following, we shall often assume the following (not so restrictive) condition on our (symmetric) monoidal ∞-categories.

Definition 3.4.14. Let $q\colon \mathscr{C}^{\otimes} \to \mathscr{O}^{\otimes}$ be an $\mathscr{O}$-monoidal ∞-category. We say that q is *compatible with colimits separately in each variable* if for every multilinear map $\varphi \in \mathrm{Mul}_{\mathscr{O}}\left(\{X_i\}_{1\leqslant i\leqslant n}, Y\right)$, the induced ∞-functor

$$\otimes_{\varphi}\colon \prod_{1\leqslant i\leqslant n} \mathscr{C}_X \longrightarrow \mathscr{C}_Y$$

commutes with colimits separately in each variable.

Example 3.4.15. If $\mathscr{O}^{\otimes} = \mathrm{Comm}^{\otimes}$ or $\mathscr{O}^{\otimes} = \mathrm{Assoc}^{\otimes}$, then we have only one object (namely, $\mathfrak{a}$ in the notation of Examples 3.4.7.(1) and 3.4.7.(3)) and $\mathscr{C} := \mathscr{C}_{\mathfrak{a}}$ for any (symmetric) monoidal ∞-category $\mathscr{C}^{\otimes}$. Therefore, in this case Definition 3.4.14 means *precisely* that for every $n \geqslant 2$, for every $i \in [1,n]$, for every $(n-1)$-uple of objects $\underline{X} := \left\{X_1 \ldots \widehat{X}_i, \ldots X_n\right\}$ of $\mathscr{C}$ and for every ∞-functor $\iota_{i,\underline{X}}\colon \mathscr{C} \to \mathscr{C}^{\times n}$ described informally as

$$Y \mapsto (X_1, \ldots, X_{i-1}, Y, X_{i+1}, \ldots X_n),$$

the composition

$$\mathscr{C} \xrightarrow{\iota_{i,\underline{X}}} \mathscr{C}^{\times n} \xrightarrow{\otimes} \mathscr{C}$$

[4] This relies on a definition of ∞-*cooperads*: we are not interested in doing it here, but we are confident that at this point it is believable enough that theory is somewhat dual to this story – but with some subtleties! See for example [FG12, Section 3.5].

commutes with colimits. For this reason, in this case we shall equivalently say that the monoidal structure of $\mathscr{C}$ *commutes with colimits separately in each variable*.

Proposition 3.4.16 (General features of ∞-categories of algebras, [Lur17, Chapter 3])**.** *Let $q\colon \mathscr{C}^{\otimes} \to \mathrm{Comm}^{\otimes}$ be a symmetric monoidal ∞-category, and suppose that the monoidal structure is compatible with colimits separately in each variable. Let $\mathscr{O}^{\otimes}$ be an ∞-operad which is small as an ∞-category, and consider the ∞-category $\mathrm{Alg}_{\mathscr{O}}(\mathscr{C})$ of $\mathscr{O}$-algebras in $\mathscr{C}^{\otimes}$.*

(1) If $\mathscr{C} := \mathscr{C}^{\otimes}_{\langle 1\rangle}$ is presentable, then $\mathrm{Alg}_{\mathscr{O}}(\mathscr{C})$ is presentable ([Lur17, Corollary 3.2.3.5]).

(2) Let $\mathrm{ev}_X\colon \mathrm{Alg}_{\mathscr{O}}(\mathscr{C}) \to \mathscr{C}_X$ be the ∞-functor that evaluates an $\mathscr{O}$-algebra at the object X of $\mathscr{O}^{\otimes}$ (if $\mathscr{O}$ has only one object, then this is just the forgetful ∞-functor that forgets the algebra structure on an object of $\mathscr{C}$). A diagram $K \to \mathrm{Alg}_{\mathscr{O}}(\mathscr{C})$ admits a limit if and only if the composition

$$K \longrightarrow \mathrm{Alg}_{\mathscr{O}}(\mathscr{C}) \xrightarrow{\mathrm{ev}_X} \mathscr{C}$$

admits a limit for every $X \in \mathscr{O}$. If this is the case, then $K \to \mathrm{Alg}_{\mathscr{O}}(\mathscr{C})$ admits a limit which is preserved by the evaluation ∞-functors ([Lur17, Corollary 3.2.2.5]).

(3) Let X be an object in $\mathscr{O}$ and let C be an object in $\mathscr{C}_X$ (if $\mathscr{O}$ has only one object, this is just an object of $\mathscr{C}$). Then there exists an $\mathscr{O}$-algebra $\mathrm{Free}_{\mathscr{O}}(X)$ and a map $X \to \mathrm{Free}_{\mathscr{O}}(X)$ which exhibits $\mathrm{Free}_{\mathscr{O}}(X)$ as the free $\mathscr{O}$-algebra generated by X ([Lur17, Proposition 3.1.3.13]). If $\mathscr{O}$ has only one object, then this yields an adjunction

$$\mathrm{Free}_{\mathscr{O}}\colon \mathscr{C} \rightleftarrows \mathrm{Alg}_{\mathscr{O}}(\mathscr{C})\colon \mathrm{oblv}_{\mathscr{O}}\,.$$

In particular, if $\mathscr{O}^{\otimes}$ is the associative or commutative ∞-operad, all the above applies. In the latter case, the free commutative algebra ∞-functor is given by the assignation

$$X \mapsto \bigoplus_{n\geqslant 0} \mathrm{Sym}^n_{\mathscr{C}}(X) := \bigoplus_{n\geqslant 0} \left[X^{\otimes n}/\Sigma_n\right],$$

where Σ_n is the symmetric group on n letters which acts on $X^{\otimes n}$ by swapping factors.

(4) If $\mathscr{O}^{\otimes} = \mathrm{Comm}^{\otimes}$, then coproducts in $\mathrm{CAlg}(\mathscr{C}) := \mathrm{Alg}_{\mathrm{Comm}}(\mathscr{C})$ are computed by the underlying tensor product in $\mathscr{C}$ ([Lur17, Proposition 3.2.4.7]).

Warning 3.4.17. The notations in Proposition 3.4.16 can be misleading. If we work in the stable derived ∞-category of a commutative ring $\mathbb{k}$ equipped with the usual (derived) tensor product of chain complexes, then the free commutative algebra object generated by a chain complex $M_\bullet$ in general *does not agree* with the polynomial algebra on some bifibrant resolution of $M_\bullet$, unless $\mathbb{k}$ is a $\mathbb{Q}$-algebra. For example, if $\mathbb{k} = \mathbb{F}_p$ and $M := \mathbb{F}_p[0]$, then the underived symmetric $\mathbb{F}_p$-algebra over $\mathbb{F}_p$ is $\mathbb{F}_p[t]$ and so we have a natural map of $\mathbb{E}_\infty$-algebras

$$\mathrm{Free}_{\mathrm{CAlg}}\left(\mathbb{F}_p\right) \longrightarrow \mathbb{F}_p[t].$$

However, this map is *not* an equivalence. Indeed, $\mathbb{F}_p[t]$ is a discrete commutative ring which is *flat* as a derived $\mathbb{F}_p$-algebra: this means that $\mathrm{H}_0(\mathbb{F}_p[t])$ is flat over $\mathrm{H}_0(\mathbb{F}_p)$, and that the homology of degree n of $\mathbb{F}_p[t]$ is obtained by the one of $\mathbb{F}_p$ by simple base change over $\mathrm{H}_0(\mathbb{F}_p[t])$; in particular it is trivial for $n > 0$. On the other hand, the homology of $\mathrm{Free}_{\mathrm{CAlg}}\left(\mathbb{F}_p\right)$ is not trivial in higher degrees: more precisely, the homology of $\mathrm{Free}_{\mathrm{CAlg}}\left(\mathbb{F}_p\right)$ in degree n is given by the homology group

$$\mathrm{H}_n\left(\mathrm{Free}_{\mathrm{CAlg}}\left(\mathbb{F}_p\right)\right) \cong \bigoplus_{k\geqslant 0} \mathrm{H}_n\left(\Sigma_k, \mathbb{F}_p\right)$$

which remembers the information concerning power operations on mod p cohomology. This heuristically means that in the homotopical setting objects in characteristic p or in mixed characteristic retain some information of *topological* nature.

Construction 3.4.18 (Modules over associative algebras). Until this point, we presented only ∞-operads with a single color (i.e., such that their underlying ∞-category has only one object). One could ask for some properly colored ∞-operad: it turns out that some of the most fundamental examples of our theory are really *colored* ∞-examples. Indeed, we have three ∞-operads $\mathscr{LM}^{\otimes}$, $\mathscr{RM}^{\otimes}$ and $\mathscr{BM}^{\otimes}$, parametrizing left modules, right modules and bimodules, respectively. Their definitions can be found in [Lur17, Sections 4.2.1 and 4.3.1]: here, we describe in full details *only* the structure of $\mathscr{LM}^{\otimes}$.

(1) We have two colors $\mathfrak{a}$ and $\mathfrak{m}$, which must be thought of as objects parametrizing an associative algebra and a module over it.
(2) The set of multilinear maps in $\mathscr{LM}^{\otimes}$ is described as follows. Let $\{X_i\}_{1\leqslant i\leqslant n}$ be a collection of n objects in $\mathscr{LM}^{\otimes}$.
   a. If all X_i's are $\mathfrak{a}$, we define $\mathrm{Mul}_{\mathscr{LM}}\left(\{X_i\}_{1\leqslant i\leqslant n}, \mathfrak{a}\right)$ to be the set of linear orderings of the set $\{1,\dots,n\}$.
   b. If all X_j's are $\mathfrak{a}$ except for one $X_{\overline{j}} = \mathfrak{m}$, we define $\mathrm{Mul}_{\mathscr{LM}}\left(\{X_i\}_{1\leqslant i\leqslant n}, \mathfrak{m}\right)$ to be the set of linear orderings on $\{1,\dots,n\}$ such that $\overline{n}$ is the maximum with respect to such orderings. This means that, if we think of a linear ordering on $\{1,\dots n\}$ as a totally ordered set $\{i_1 < \dots < i_n\}$, corresponding to the inverse of the permutation $\sigma \in \Sigma_n$ sending $j \mapsto i_j$, then we want i_n to be $\overline{j}$.
   c. In any other case, $\mathrm{Mul}_{\mathscr{LM}}\left(\{X_i\}_{1\leqslant i\leqslant n}, Y\right)$ is empty.
(3) The composition is given by composition of linear orderings.

By similar arguments to the ones provided in Example 3.4.13.(2), we see that a $\mathscr{LM}$-algebra in a monoidal ∞-category $\mathscr{C}^{\otimes}$ selects two objects A and M, such that A is an associative algebra (indeed, the n-linear maps from $\mathfrak{a}^{\otimes n}$ to $\mathfrak{a}$ in $\mathscr{LM}^{\otimes}$ are precisely the same as the n-linear maps from $\mathfrak{a}^{\otimes n}$ to $\mathfrak{a}$ in $\mathrm{Assoc}^{\otimes}$) and M is a left module over A: this justifies the notation $\mathrm{LMod}(\mathscr{C})$ for the ∞-category $\mathrm{Alg}_{\mathscr{LM}}(\mathscr{C})$. The ∞-operad $\mathscr{RM}^{\otimes}$ is defined in a very similar way, and the ∞-category $\mathscr{RM}$-algebras in a monoidal ∞-category $\mathscr{C}$ will be denoted by $\mathrm{RMod}(\mathscr{C})$. Finally; the ∞-operad $\mathscr{BM}^{\otimes}$ intertwines these two definitions by specifying three colors $\mathfrak{a}$, $\mathfrak{a}'$, and $\mathfrak{m}$, such that $\mathfrak{a}$ and $\mathfrak{a}'$ are both associative algebras, and $\mathfrak{m}$ is a left $\mathfrak{a}$-module and a right $\mathfrak{a}'$-module in a compatible way: for this reason, we shall denote the ∞-category of $\mathscr{BM}$-algebras in a monoidal ∞-category $\mathscr{C}$ as $\mathrm{BMod}(\mathscr{C})$.

Remark 3.4.19. For any symmetric monoidal ∞-category $\mathscr{C}$, we have a diagram of ∞-categories

$$\begin{tikzcd}
 & \mathrm{CAlg}(\mathscr{C}) \arrow[d, "\mathrm{oblv}_{\mathrm{Comm}}"'] & \\
 & \mathrm{Alg}(\mathscr{C}) \arrow[d] & \\
\mathrm{LMod}(\mathscr{C}) \arrow[ur, bend left, "\mathrm{oblv}_{\mathrm{LMod}}"] & \mathrm{BMod}(\mathscr{C}) \arrow[l, "\mathrm{oblv}_{\mathrm{RMod}}"] \arrow[r, "\mathrm{oblv}_{\mathrm{LMod}}"'] \arrow[u, shift left=2, "\mathrm{oblv}^{\mathrm{L}}_{\mathrm{BMod}}"] \arrow[u, shift right=2, "\mathrm{oblv}^{\mathrm{R}}_{\mathrm{BMod}}"'] & \mathrm{RMod}(\mathscr{C}) \arrow[ul, bend right, "\mathrm{oblv}_{\mathrm{RMod}}"']
\end{tikzcd}$$

where each ∞-functor is described as follows.

(1) The ∞-functor $\mathrm{oblv}_{\mathrm{Comm}}\,\mathrm{CAlg}(\mathscr{C}) \to \mathrm{Alg}(\mathscr{C})$ is induced by precomposing a map of ∞-operads $\mathrm{Comm}^{\otimes} \to \mathscr{C}^{\otimes}$ with the fibration $\mathrm{Assoc}^{\otimes} \to \mathrm{Comm}^{\otimes}$. This ∞-functor simply takes a commutative algebra A in $\mathscr{C}$ and forgets its commutative structure to a merely associative one.

(2) The ∞-functors $\mathrm{oblv}_{\mathrm{LMod}} \colon \mathrm{LMod}(\mathscr{C}) \to \mathrm{Alg}(\mathscr{C})$ and $\mathrm{oblv}_{\mathrm{RMod}} \colon \mathrm{RMod}(\mathscr{C}) \to \mathrm{Alg}(\mathscr{C})$ are again induced by precomposing maps of ∞-operads $\mathscr{LM}^{\otimes} \to \mathscr{C}^{\otimes}$ annd $\mathscr{RM}^{\otimes} \to \mathscr{C}^{\otimes}$ with the natural inclusions $\mathrm{Assoc}^{\otimes} \hookrightarrow \mathscr{LM}^{\otimes}$ and $\mathrm{Assoc}^{\otimes} \hookrightarrow \mathscr{RM}^{\otimes}$, respectively. They forget the datum of the module object of $\mathscr{C}$ retaining only the associative algebra. Similar arguments hold for the ∞-functors $\mathrm{oblv}^{\mathrm{L}}_{\mathrm{BMod}}$ and $\mathrm{oblv}^{\mathrm{R}}_{\mathrm{BMod}}$, which for a fixed bimodule object (A, M, A') remember either the associative algebra A' or the associative algebra A, respectively. It is clear that if A and A' are commutative then these forgetful ∞-functors actually factor through $\mathrm{CAlg}(\mathscr{C})$.

(3) The ∞-functors $\mathrm{oblv}_{\mathrm{RMod}} \colon \mathrm{BMod}(\mathscr{C}) \to \mathrm{LMod}(\mathscr{C})$ and $\mathrm{oblv}_{\mathrm{LMod}} \colon \mathrm{BMod}(\mathscr{C}) \to \mathrm{LMod}(\mathscr{C})$ are induced by precomposing a map of ∞-operads $\mathscr{BM}^{\otimes} \to \mathscr{C}^{\otimes}$ with the natural inclusions $\mathscr{LM}^{\otimes} \hookrightarrow \mathscr{BM}^{\otimes}$ and $\mathscr{RM}^{\otimes} \hookrightarrow \mathscr{BM}^{\otimes}$. Given a bimodule (A, M, A') in $\mathscr{C}$, the first ∞-functor forgets the datum of A' and the second forgets the datum of A, producing the left module (A, M) and the right module (M, A'), respectively. The choice of notation for these ∞-functors could appear a bit lazy at first glance: *a priori*, it may seem that it clashes with the notations for the ∞-functors $\mathrm{oblv}_{\mathrm{LMod}}$ and $\mathrm{oblv}_{\mathrm{RMod}}$ that we defined in the previous point. However, Remark 3.4.22 guarantees that this is not the case: we are indeed forgetting some either left or right module structure.

(4) The ∞-functor $\mathrm{Alg}(\mathscr{C}) \to \mathrm{BMod}(\mathscr{C})$ is induced by the forgetful map of ∞-operads $\mathscr{BM}^{\otimes} \to \mathrm{Assoc}^{\otimes}$ described by sending the colors $\mathfrak{a}$, $\mathfrak{a}'$ and $\mathfrak{m}$ to the single color $\mathfrak{a}$ in $\mathrm{Assoc}^{\otimes}$, and by sending the multilinear map $\mathfrak{a}^{\otimes i} \otimes \mathfrak{m} \otimes \mathfrak{a}'^{\otimes n-i-1} \to \mathfrak{m}$ corresponding to some suitable linear ordering on the set $\{1, \dots n\}$ to the multilinear map $\mathfrak{a}^{\otimes n} \to \mathfrak{a}$ corresponding to the same linear ordering in $\mathrm{Assoc}^{\otimes}$. In particular, such ∞-functors takes an associative algebra A and considers it as a bimodule over itself.

Definition 3.4.20. If we fix an associative algebra A in $\mathscr{C}$, the fiber of A under the ∞-functors $\mathrm{oblv}_{\mathrm{LMod}} \colon \mathrm{LMod}(\mathscr{C}) \to \mathrm{Alg}(\mathscr{C})$ and $\mathrm{oblv}_{\mathrm{RMod}} \colon \mathrm{RMod}(\mathscr{C}) \to \mathrm{Alg}(\mathscr{C})$ yield ∞-categories $\mathrm{LMod}_A(\mathscr{C})$ and $\mathrm{RMod}_A(\mathscr{C})$, the *$\infty$-categories of left* or *right A-modules*, respectively. Similarly, if we fix associative algebras A and B in $\mathscr{C}$, the fiber ${}_A\mathrm{BMod}_B(\mathscr{C})$ of (A, B) under the forgetful

∞-functor

$$\mathrm{oblv}^{\mathrm{L}}_{\mathrm{BMod}} \times \mathrm{oblv}^{\mathrm{R}}_{\mathrm{BMod}} \colon \mathrm{BMod}(\mathscr{C}) \longrightarrow \mathrm{Alg}(\mathscr{C}) \times \mathrm{Alg}(\mathscr{C})$$

yields the *∞-category of (A, B)-bimodules*.

Proposition 3.4.21 (Useful properties of ∞-categories of modules, [Lur17, Sections 4.2 and 4.3]). *Let $\mathscr{C}^{\otimes}$ be a symmetric monoidal ∞-category and let A be an associative algebra in $\mathscr{C}$.*

(1) If $\mathscr{C}$ is presentable, then $\mathrm{LMod}_A(\mathscr{C})$ and $\mathrm{RMod}_A(\mathscr{C})$ are presentable.

(2) Limits in $\mathrm{LMod}_A(\mathscr{C})$ and $\mathrm{RMod}_A(\mathscr{C})$ are always detected by the forgetful ∞-functor to $\mathscr{C}$. If the monoidal structure of $\mathscr{C}$ commutes with colimits separately in each variable, the same statement holds for colimits. In particular, if $\mathscr{C}$ is stable then also $\mathrm{LMod}_A(\mathscr{C})$ and $\mathrm{RMod}_A(\mathscr{C})$ are stable.

(3) The ∞-categories $\mathrm{LMod}_A(\mathscr{C})$ and $\mathrm{RMod}_A(\mathscr{C})$ are respectively right and left tensored over $\mathscr{C}$, i.e., we have actions

$$\mathrm{LMod}_A(\mathscr{C}) \times \mathscr{C} \longrightarrow \mathrm{LMod}_A(\mathscr{C})$$

and

$$\mathscr{C} \times \mathrm{RMod}_A(\mathscr{C}) \longrightarrow \mathrm{RMod}_A(\mathscr{C})$$

given by the underlying tensor product in $\mathscr{C}$. If the monoidal structure of $\mathscr{C}$ commutes with colimits separately in each variable, then both actions commute with colimits separately in each variable.

Remark 3.4.22. Let $\mathscr{C}^{\otimes}$ be a symmetric monoidal ∞-category and let A and B be associative algebras of $\mathscr{C}$. Thanks to the actions described in Proposition 3.4.21.(3), we can consider right B-modules with a left action of A and left A-modules with a right action of B, and one has equivalences of ∞-categories

$${}_A\mathrm{BMod}_B(\mathscr{C}) \simeq \mathrm{LMod}_A(\mathrm{RMod}_B(\mathscr{C})) \simeq \mathrm{RMod}_B(\mathrm{LMod}_A(\mathscr{C}))$$

thanks to [Lur17, Theorem 4.3.2.7]. In particular, the content of Proposition 3.4.21.(1) and the statement on limits of Proposition 3.4.21.(2) apply also to ${}_A\mathrm{BMod}_B(\mathscr{C})$. If moreover the symmetric monoidal structure of $\mathscr{C}$ commutes with colimits separately in each variable, then the statement on colimits of Proposition 3.4.21.(2) applies to ${}_A\mathrm{BMod}_B(\mathscr{C})$ as well.

Warning 3.4.23. If A is a *commutative* algebra, then there is an equivalence of ∞-categories

$$\mathrm{LMod}_A(\mathscr{C}) \simeq \mathrm{RMod}_A(\mathscr{C}) =: \mathrm{Mod}_A(\mathscr{C})$$

but $\mathrm{BMod}_A(\mathscr{C})$ can be *very* different from both $\mathrm{LMod}_A(\mathscr{C})$ and $\mathrm{RMod}_A(\mathscr{C})$, even if A is commutative. Indeed, $\mathrm{BMod}_A(\mathscr{C})$ is equivalent to the ∞-category of left $A \otimes A^{\mathrm{rev}}$-modules, where $\otimes$ is the monoidal structure on $\mathscr{C}$ and A^{rev} is A itself seen as an associative algebra via the "opposite" multiplication. (To be more formal, if A corresponds to the map of ∞-operads $f : \mathrm{Assoc}^{\otimes} \to \mathscr{C}^{\otimes}$, then A^{rev} corresponds to the map of ∞-operads obtained by precomposing f with the autoinvolution of $\mathrm{Assoc}^{\otimes}$, sending a linear ordering of the set $\{1, \ldots, n\}$ to its opposite linear order.) In particular, if A is commutative and $A \otimes A \not\simeq A$, then the ∞-category of A-bimodules is often different from the ∞-category of A-modules. In any case, we do have a forgetful ∞-functor

$$\mathrm{Mod}_A(\mathscr{C}) \longrightarrow \mathrm{BMod}_A(\mathscr{C}).$$

Remark 3.4.24. Considering modules over associative algebras is actually a functorial operation, in the following sense. The ∞-functors

$$\mathrm{oblv}_{\mathrm{LMod}}\colon \mathrm{LMod}(\mathscr{C}) \longrightarrow \mathrm{Alg}(\mathscr{C}) \quad \text{and} \quad \mathrm{oblv}_{\mathrm{RMod}}\colon \mathrm{RMod}(\mathscr{C}) \longrightarrow \mathrm{Alg}(\mathscr{C})$$

are both Cartesian and coCartesian fibrations, hence the Grothendieck construction of Theorem 2.7.13 provides ∞-functors

$$\mathrm{LMod}\_(\mathscr{C})\colon \mathrm{Alg}(\mathscr{C}) \longrightarrow \widehat{\mathrm{Cat}}_\infty \quad \text{and} \quad \mathrm{RMod}\_(\mathscr{C})\colon \mathrm{Alg}(\mathscr{C}) \longrightarrow \widehat{\mathrm{Cat}}_\infty,$$

which assign to any associative algebra A its ∞-category of either left or right modules, in an ∞-functorial way. This association can be regarded either as a covariant one (given a morphism $f : A \to B$ the induced ∞-functor $\mathrm{LMod}_A(\mathscr{C}) \to \mathrm{LMod}_B(\mathscr{C})$ is the base change of a left A-module along f) or as a contravariant one (given a morphism $f : A \to B$, the induced ∞-functor $\mathrm{LMod}_B(\mathscr{C}) \to \mathrm{LMod}_A(\mathscr{C})$ forgets the left B-module structure along f). Similarly, using cleverly Remark 3.4.22, we obtain an ∞-functor

$$\_\mathrm{BMod}\_(\mathscr{C})\colon \mathrm{Alg}(\mathscr{C})^{\mathrm{op}} \times \mathrm{Alg}(\mathscr{C}) \longrightarrow \widehat{\mathrm{Cat}}_\infty,$$

which sends a pair (A,A') of associative algebras to the ∞-category ${}_A\mathrm{BMod}_{A'}(\mathscr{C})$ of (A,A')-*bimodules*; if $A = A'$, we shall simply denote such ∞-category as $\mathrm{BMod}_A(\mathscr{C})$. Again, this association is both covariant and contravariant in both arguments.

We now present a class of ∞-operads via a *topologically enriched* model: the *little cubes ∞-operads* $\mathbb{E}_k$. This is a particularly relevant class for our scopes because of the following reasons.

(1) For $k = 0$, the ∞-operad $\mathbb{E}_0$ classifies objects pointed by the monoidal unit of $\mathscr{C}$.
(2) For $k = 1$, the ∞-operad $\mathbb{E}_1$ is equivalent to the operad Assoc.
(3) For any $k \geqslant 2$, the ∞-operad $\mathbb{E}_k$ classifies algebras which are "k-fold commutative". In particular, we have a filtered diagram of ∞-operads

$$\mathbb{E}_0 \hookrightarrow \mathbb{E}_1 \hookrightarrow \mathbb{E}_2 \hookrightarrow \dots \hookrightarrow \mathbb{E}_k \hookrightarrow \dots$$

whose colimit $\mathbb{E}_\infty$ is equivalent to the ∞-operad Comm.

Construction 3.4.25 (Little cubes ∞-operad, [Lur17, Chapter 5])**.** For any $k \geqslant 0$, let us denote by $\square^k$ the standard k-dimensional cube $[-1,1]^k \subseteq \mathbb{R}^k$. We say that a *rectilinear embedding* $f : \square^k \to \square^k$ is a map described in each component as $f_i(\underline{x}) = a_i x_i + b_i$, for some real constants $a_i > 0$ and b_i. For any finite set S considered as a discrete topological space, we say that a map $f : \square^k \times S \to \square^k$ is a rectilinear embedding if it is an open embedding and $f_s\colon \square^k \times \{s\} \cong \square^k \to \square^k$ is rectilinear in the previous sense for any $s \in S$. For any finite set S, the set

$$\mathrm{Rect}\big(\square^k \times S, \square^k\big) := \big\{f : \square^k \times S \to \square^k \mid f \text{ is a rectilinear embedding}\big\}$$

can be interpreted as a subset of $\big(\mathbb{R}^{2k}\big)^S$, and it inherits a natural subspace topology. For any $k \geqslant 0$ we define a topologically enriched category ${}^{\mathrm{t}}\mathbb{E}_k^\otimes$ as follows.

(1) It has all finite pointed sets $\langle n \rangle \in \mathrm{Fin}_*$ as objects.

(2) The topological space $\mathrm{Hom}_{{}^{\mathrm{t}}\mathbb{E}_k^{\otimes}}(\langle n\rangle, \langle m\rangle)$ is the set of $(n+1)$-uples $\big(\alpha, \langle f_j\rangle_{j\in\langle \mathring{m}\rangle}\big)$, where $\alpha\colon \langle n\rangle \to \langle m\rangle$ is a map of pointed sets and $f_j\colon \square^k \times \alpha^{-1}(j) \to \square^k$ is a rectilinear embedding. The topology is induced by the presentation

$$\mathrm{Hom}_{{}^{\mathrm{t}}\mathbb{E}_k^{\otimes}}(\langle n\rangle, \langle m\rangle) := \coprod_{\alpha\colon \langle n\rangle\to\langle m\rangle} \prod_{1\leqslant j\leqslant m} \mathrm{Rect}\big(\square^k \times \alpha^{-1}(j), \square^k\big).$$

We have an associated ∞-category $\mathbb{E}_k^{\otimes}$, and the natural forgetful ∞-functor $\mathbb{E}_k^{\otimes} \to \mathrm{Fin}_*$ turns $\mathbb{E}_k^{\otimes}$ into an ∞-operad, whose underlying ∞-category is the ∞-category $\mathbb{E}_k := \mathbb{E}_k^{\otimes} \times_{\mathrm{Fin}_*} \{\langle 1\rangle\}$ consisting of only one object, which is $\langle 1\rangle$ itself, with the following spaces of maps. The associated unpointed set $\langle \mathring{1}\rangle$ of $\langle 1\rangle$ is the point $\{1\}$. We have now two possible pointed maps from $\langle 1\rangle$ to itself: the non-active map α (i.e., the one that factors through the base point $\{*\}$) and the active map β (i.e., the identity). The map α has an attached topological space of empty rectilinear embeddings, so this is simply a point. For the case of the active map β, we have instead a proper topological space of rectilinear embeddings from $\square^k$ to itself. This space is homeomorphic to a complex polytope in $\mathbb{R}^{2k}$, hence it is contractible.
In particular, for any $k \geqslant 0$, an $\mathbb{E}_k$-algebra in a symmetric monoidal ∞-category $\mathscr{C}^{\otimes}$ is the datum of an object X, with no non-trivial maps $X \to X$. The space of multilinear maps from $\langle n\rangle$ to $\langle 1\rangle$ encodes a multiplication $X^{\otimes n} \to X$, with some rules that have to be satisfied up to coherent homotopy. The space of multilinear maps from $\langle 0\rangle$ to $\langle 1\rangle$ encodes the notion of a pointing $\mathbb{1}_{\mathscr{C}} \to X$ – i.e., the multiplication defined by the ∞-operad $\mathbb{E}_k$ is actually unitary for any k. We shall see how this works in fuller detail in the following example.

Example 3.4.26. Let us study $\mathbb{E}_k^{\otimes}$ via this presentation in greater detail for some relevant k's.

(1) For $k = 0$, $\square^k \cong \{*\}$ is just a point. The description of the underlying ∞-category of $\mathbb{E}_0$ is the same as the one shown in Construction 3.4.25, so let us look at the space of multilinear maps $\mathrm{Hom}_{{}^{\mathrm{t}}\mathbb{E}_k^{\otimes}}(\langle n\rangle, \langle m\rangle)$. Suppose that $\alpha\colon \langle n\rangle \to \langle m\rangle$ is such that there exists an element $j \in \langle \mathring{m}\rangle$ such that $\left|\alpha^{-1}(j)\right| > 1$: then

$$\prod_{1\leqslant k\leqslant m} \mathrm{Rect}\big(\{*\} \times \alpha^{-1}(k), \{*\}\big)$$

would be a product of topological spaces with one factor corresponding to rectilinear embeddings $\{*\} \times \alpha^{-1}(j) \to \{*\}$. Since a rectilinear embedding is in particular an open embedding, this space is empty, hence the whole product is empty. On the other hand, if for any $j \in \langle \mathring{m}\rangle$ the fiber is at most a single point, then

$$\prod_{1\leqslant k\leqslant m} \mathrm{Rect}\big(\{*\} \times \alpha^{-1}(k), \{*\}\big)$$

is a product of points, hence a point itself.
This tells us that the space $\mathrm{Mul}_{\mathbb{E}_0^{\otimes}}(\langle n\rangle, \langle 1\rangle)$ is empty except for $n = 0$ and for $n = 1$.This means that an $\mathbb{E}_0$-algebra X in a symmetric monoidal ∞-category $\mathscr{C}$ consists only of the datum of a pointing $\mathbb{1}_{\mathscr{C}} \to X$ (corresponding to the multilinear map $\langle 0\rangle \to \langle 1\rangle$), with the identity as the only linear map $X \to X$. We have a canonical equivalence of ∞-categories

$$\mathscr{C}_{\mathbb{1}_{\mathscr{C}}/} \simeq \mathrm{Alg}_{\mathbb{E}_0}(\mathscr{C}).$$

(2) For $k = 1$, $\mathbb{E}_1^{\otimes}$ is an ∞-operad on a single pointed object $\mathbb{I} := [-1,1]$. For any pointed map of finite sets $\alpha\colon \langle n\rangle \to \langle m\rangle$ and any $j \in \langle \mathring{m}\rangle$, a rectilinear embedding $f : \mathbb{I}\times\alpha^{-1}(j) \to \mathbb{I}$ induces a linear ordering on $\alpha^{-1}(j)$ as follows: for any h and k in $\alpha^{-1}(j)$ we say that $h \preccurlyeq k$ if $f(t,h) \leqslant f(t',k)$ for any couple of points t and t' in $\mathbb{I}$. More precisely, if $\alpha^{-1}(j) = \{j_1,\ldots,j_n\}$ then

$$\mathbb{I}\times\alpha^{-1}(j) \cong \coprod_{1\leqslant k\leqslant n} \mathbb{I}(k),$$

where (k) denotes that we are labeling the k-th interval in the sequence. Considering $\mathbb{I}$ with its standard linear order inherited by $\mathbb{R}$, the linear order on $\alpha^{-1}(j)$ is given by *remembering the order* with which we embed the intervals $\mathbb{I}(k)$ inside $\mathbb{I}$. Viceversa, knowing a linear order $\preccurlyeq$ on $\alpha^{-1}(j)$ produces a contractible collection of rectilinear embeddings of $\mathbb{I}\times\alpha^{-1}(j) \to \mathbb{I}$ subject to the condition that $\mathbb{I}(h)$ is embedded to the left of $\mathbb{I}(k)$ if $h \preccurlyeq k$. In particular, $\mathbb{E}_1^{\otimes}$ is equivalent as an ∞-operad to $\mathrm{Assoc}^{\otimes}$: the fact that the operation is associative but not commutative is encoded in the fact that, in the 1-dimensional square, we do not have enough space to swap two embedded intervals in a continuous fashion. In particular, for any symmetric monoidal ∞-category $\mathscr{C}$ we have a canonical equivalence

$$\mathrm{Alg}_{\mathbb{E}_1}(\mathscr{C}) \simeq \mathrm{Alg}(\mathscr{C}).$$

(3) For $k \geqslant 2$, let us study the space of multilinear maps lying over the active map $\beta\colon \langle 2\rangle \to \langle 1\rangle$. The space of rectilinear embeddings in this case is a conctractible space of configurations of two k-disks inside $\square^k$, which parametrizes a non-trivial multiplication $X\otimes X \to X$. Notice now that for any $n \geqslant 3$, every space of configurations of n possible k-disks inside $\square^k$ is contractible, since we can rotate and move them inside $\square^k$ in order to change from a configuration to another: so, the multiplication of an $\mathbb{E}_{\geqslant 2}$-algebra X is indeed always commutative. Yet, following the principle that ∞-category theory *remembers the datum of the homotopy coherencies*, we have now that the space of homotopies between configurations is way different changing the integer k: increasing the topological dimension, we have more degrees of freedom for moving our smaller k-disks inside a larger k-disk. More formally, the space of rectilinear embeddings $\square^k \times I \to \square^k$ is $(k-1)$-connected, i.e., it has trivial homotopy groups for $n \leqslant k-2$ ([Lur17, Proposition 5.1.4.4]). Notice that this matches our computation for the case $k = 0$ (every space is (-2)-connected, but it is (-1)-connected only if it is not empty) and $k = 1$ (we have always non-empty spaces, hence (-1)-connected, which are connected only when I is just a singleton).

(4) For any k, the ∞-operad $\mathbb{E}_k^{\otimes}$ embeds in the ∞-operad $\mathbb{E}_{k+1}^{\otimes}$ by taking the product of a rectilinear embedding $\square^k \times I \to \square^k$ with the identity on another copy of the interval $\mathbb{I}$. This produces a tower of ∞-operads whose colimit

$$\mathbb{E}_\infty^{\otimes} := \operatorname*{colim}_{k\geqslant 0} \mathbb{E}_k^{\otimes}$$

is characterized by the fact that the space of rectilinear embeddings of the "infinite-dimensional disk" to itself is fully contractible (because it has to be $(k-1)$-connective for any integer k). In particular, $\mathbb{E}_\infty^\otimes \simeq \mathrm{Comm}^\otimes$ and an $\mathbb{E}_\infty$-algebra in a symmetric monoidal ∞-category $\mathscr{C}^\otimes$ is simply a *fully commutative algebra*, i.e., we have a natural equivalence of ∞-categories

$$\mathrm{CAlg}(\mathscr{C}) \simeq \mathrm{Alg}_{\mathbb{E}_\infty}(\mathscr{C}).$$

Warning 3.4.27. For any $0 \leqslant k < h \leqslant \infty$, an $\mathbb{E}_h$-algebra in a symmetric monoidal ∞-category – thought as map of ∞-operads $f : \mathbb{E}_h^\otimes \to \mathscr{C}^\otimes$ over $\mathrm{Comm}^\otimes$ – can be turned into an $\mathbb{E}_k$-algebra by precomposing f with the inclusion of ∞-operads $\mathbb{E}_k^\otimes \hookrightarrow \mathbb{E}_h^\otimes$. This produces a chain of forgetful ∞-functors

$$\mathrm{CAlg}(\mathscr{C}) \xrightarrow{\mathrm{oblv}_{\mathrm{CAlg}}} \ldots \longrightarrow \mathrm{Alg}_{\mathbb{E}_{k+1}}(\mathscr{C}) \xrightarrow{\mathrm{oblv}_{\mathbb{E}_{k+1}}} \mathrm{Alg}_{\mathbb{E}_k}(\mathscr{C}) \xrightarrow{\mathrm{oblv}_{\mathbb{E}_k}} \mathrm{Alg}_{\mathbb{E}_{k-1}}(\mathscr{C}) \longrightarrow \ldots \longrightarrow \mathrm{Alg}_{\mathbb{E}_0}(\mathscr{C}) \simeq \mathscr{C}_{\mathbb{1}_\mathscr{C}/}.$$

These ∞-functors are *not* fully faithful. Notice that, in ordinary algebra, forgetting the commutativity of a monoid, a group or a ring is indeed a fully faithful operation: maps of abelian groups and commutative rings and monoids are simply maps of groups, rings and monoids, and the commutativity is preserved automatically because in this setting *commutativity is a property*. On the other hand, in the homotopical world a map of $\mathbb{E}_k$-algebras $f : A \to B$ is a map of the underlying associative algebras together with the *datum* of a transformation of the simplices testifying to the $\mathbb{E}_k$-commutativity of A into simplices testifying to the $\mathbb{E}_k$-commutativity of B. Differently from what happened in the discrete case, this assignation is not determined uniquely from the map of the underlying associative algebras.

We conclude this section with a fundamental theorem concerning the little disk operad.

Theorem 3.4.28 (Dunn Additivity, [Lur17, Theorem 5.1.2.2])**.** *Let $k, k' \geqslant 0$ be integers. Then for any monoidal ∞-category $\mathscr{C}$ there is an equivalence of ∞-categories*

$$\mathrm{Alg}_{\mathbb{E}_{k+k'}}(\mathscr{C}) \simeq \mathrm{Alg}_{\mathbb{E}_k}\big(\mathrm{Alg}_{\mathbb{E}_{k'}}(\mathscr{C})\big),$$

where $\mathrm{Alg}_{\mathbb{E}_{k'}}(\mathscr{C})$ is considered as a monoidal ∞-category via the underlying tensor product of $\mathscr{C}$ (Example 3.4.13.(5)).

Theorem 3.4.28 formalizes the idea that we were hinting at in the introduction of Section 3.3: n-fold H-spaces are spaces equipped with n different associative (but only up to homotopy!) operations, which are compatible one with the other. Indeed, with our brand new terminology, May's Theorem can be read as follows.

Theorem 3.4.29 (May's Delooping, [Lur17, Theorem 5.2.6.10])**.** *For any $k > 0$, let $\mathcal{S}_*^{\geqslant k}$ be the ∞-category spanned by $(k-1)$-connected pointed spaces. Then the iterated loop ∞-functor*

$$\Omega^k : \mathcal{S}_*^{\geqslant k} \longrightarrow \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{S})$$

is a fully faithful ∞-functor, whose essential image is spanned by the ∞-category $\mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathscr{C})$ of group-like $\mathbb{E}_k$-monoids in spaces, i.e., topological $\mathbb{E}_k$-monoids M for which $\pi_0 M$ is a group[5]*. An explicit inverse is provided by the k-fold delooping ∞-functor*

$$\mathbf{B}^k : \mathrm{Alg}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathscr{C}) \longrightarrow \mathcal{S}_*^{\geqslant k}.$$

[5]In general, a homotopy-associative monoid structure over a nice topological space only provides its set of path-connected components a monoid structure, without an inverse.

3.5. **Algebra in the stable ∞-category.** The concepts presented in Section 3.4 allow us to deepen the characterization of the ∞-category Sp of spectra, and in particular to do *algebra* in it: the theory of *spectral algebraic geometry* serves as a middle ground between homotopical algebra and homological algebra, allowing to study objects which behave like "homotopy coherent commutative rings" with the tools of algebraic geometry (e.g., by looking at derived stacks which locally look like homotopy coherent coherent rings and at their ∞-categories of quasi-coherent sheaves), while at the same time providing a more well-rounded foundation to the theory of derived functors and derived categories of schemes.

Theorem 3.5.1 ([Lur17, Corollary 4.8.2.19]). *The ∞-category of spectra* Sp *is a stable and presentable ∞-category and can be equipped with a symmetric monoidal structure provided by the smash product of spectra*

$$\wedge\colon \mathrm{Sp}\times\mathrm{Sp}\longrightarrow\mathrm{Sp}.$$

The smash product is characterized in an essentially unique way by the requirements that it commutes with colimits separately in each variable and that the sphere spectrum

$$\mathbb{S}:=\Sigma^{\infty}_{+}(\{*\})$$

acts as the unit for such monoidal structure. Moreover, for any stable and presentable ∞-category $\mathscr{C}$ equipped with a symmetric monoidal structure compatible with colimits separately in each variable, there exists an essentially unique colimit preserving and strongly monoidal ∞-functor $\mathrm{Sp}\to\mathscr{C}$.

Theorem 3.5.1 is a "stable variant" of the following statement: the ∞-category $\mathcal{S}$ of spaces with its Cartesian monoidal structure is universal among presentable ∞-categories equipped with a symmetric monoidal structure which is compatible with colimits separately in each variable. Indeed, for any presentable ∞-category $\mathscr{C}$ as such, we have an essentially unique ∞-functor

$$\mathcal{S}\longrightarrow\mathscr{C}$$

characterized by the requirement that it is strongly monoidal and that the point $\{*\}$ is sent to the unit $\mathbb{1}_{\mathscr{C}}$ for the monoidal structure of $\mathscr{C}$. If $\mathscr{C}$ is also stable, then the universal property of the stabilization of an ∞-category yields that $\mathcal{S}\to\mathscr{C}$ factors through $\mathrm{Sp}(\mathcal{S})=:\mathrm{Sp}$. This statement can be further refined as follows. First, let us remark that the ∞-category Pr^{L} of presentable ∞-categories and left adjoints between them admits a non-trivial model structure (i.e., not induced by the product of ∞-categories in $\widehat{\mathrm{Cat}}_{\infty}$).

Theorem 3.5.2 ([Lur17, Section 4.8.1]). *The ∞-category Pr^{L} of presentable ∞-categories is endowed with a closed symmetric monoidal tensor product $\otimes$, for which the unit is the ∞-category of spaces $\mathcal{S}$. If $\mathrm{Fun}^{\mathrm{R}}(\mathscr{C},\mathscr{D})$ denotes the ∞-category of limit-preserving ∞-functors between $\mathscr{C}$ and $\mathscr{D}$, then for every presentable ∞-categories $\mathscr{C}$ and $\mathscr{D}$ one has*

$$\mathscr{C}\otimes\mathscr{D}\simeq\mathrm{Fun}^{\mathrm{R}}(\mathscr{C}^{\mathrm{op}},\mathscr{D}).$$

Such monoidal structure commutes with colimits of presentable ∞-categories seprately in each variable (Definition 3.4.14), and objects in the ∞-category of $\mathbb{E}_k$-algebras in $\left(\mathrm{Pr}^{\mathrm{L}}\right)^{\otimes}$ can be identified with $\mathbb{E}_k$-monoidal presentable ∞-categories whose monoidal structure is compatible with colimits *separately in each variable. The symmetric monoidal structure on* Pr^{L} *enjoys the following universal property: any* ∞*-functor* $\mathscr{C}\times\mathscr{D}\to\mathscr{E}$ *which commutes with colimits separately in each variable factors in an essentially unique way via an* ∞*-functor* $\mathscr{C}\times\mathscr{D}\to\mathscr{C}\otimes\mathscr{D}$*.*

Theorem 3.5.3 ([Lur17, Section 4.8.2])**.** *The* ∞*-category* Sp *is an* $\mathbb{E}_\infty$*-algebra object of* Pr^{L} *endowed with the symmetric monoidal structure of Theorem 3.5.2, and the forgetful* ∞*-functor*

$$\mathrm{Mod}_{\mathrm{Sp}}(\mathrm{Pr}^{\mathrm{L}})\longrightarrow \mathrm{Pr}^{\mathrm{L}}$$

is fully faithful. Its essential image is the full sub-∞*-category* $\mathrm{Stab}^{\mathrm{pr}}_\infty\subseteq \mathrm{Pr}^{\mathrm{L}}$ *spanned by presentable and stable* ∞*-categories.*

Remark 3.5.4. Let $\mathscr{C}$ be a presentable and stable ∞-category. Then, Theorem 3.5.3 tells us that $\mathscr{C}$ is a Sp-module in Pr^{L}, hence we have an ∞-functor

$$\mathrm{Sp}\times\mathscr{C}\longrightarrow\mathscr{C}$$

which exhibits $\mathscr{C}$ as a module over Sp and which is compatible with colimits separately in each variable. In particular, for any objects C and D in $\mathscr{C}$ the contravariant ∞-functor $\mathrm{Sp}^{\mathrm{op}}\longrightarrow \mathscr{S}$ determined by the assignation $X\mapsto \mathrm{Map}_{\mathscr{C}}(X\times C, D)$ sends colimits to limits, hence it is representable by a spectrum $\underline{\mathrm{Map}}(C,D)$ in virtue of [Lur09, Proposition 5.5.2.2]. It follows that the ∞-functor

$$\mathscr{C}^{\mathrm{op}}\times\mathscr{C}\longrightarrow \mathrm{Fun}(\mathrm{Sp}^{\mathrm{op}},\mathscr{S})$$

sending a couple of objects (C,D) to the mapping space $\mathrm{Map}_{\mathscr{C}}(-\times C, D)$ lands in the essential image of the Yoneda embedding $よ\colon \mathrm{Sp}\to\mathrm{Fun}(\mathrm{Sp}^{\mathrm{op}},\mathscr{S})$, hence after composing with a homotopy inverse to $よ$ we obtain a honest ∞-functor

$$\underline{\mathrm{Map}}_{\mathscr{C}}\colon \mathscr{C}^{\mathrm{op}}\times\mathscr{C}\longrightarrow\mathrm{Sp}$$

producing a full *spectrum* of maps between objects of $\mathscr{C}$. In the following, when dealing with stable and presentable ∞-categories, we shall always consider the mapping *spectrum* of maps between objects, instead of the usual mapping *space*.

Considering $\mathbb{E}_k$-algebras for the smash product in the ∞-category of spectra yields the starting point of homotopical algebra.

Definition 3.5.5 ([Lur17, Definition 7.1.0.1])**.** Fix an element $k\in\mathbb{N}_{\geqslant 1}\cup\{\infty\}$. An $\mathbb{E}_k$*-ring spectrum* is an $\mathbb{E}_k$-algebra object in the symmetric monoidal ∞-category of spectra.

Notation 3.5.6. In the following, we shall suppress some notation and simply denote by $\mathrm{Alg}_{\mathbb{E}_k}$ the ∞-category of $\mathbb{E}_k$-ring spectra. In the case $k=1$, we shall further compactify the notations and simply denote $\mathrm{Alg}_{\mathbb{E}_1}$, the ∞*-category of associative ring spectra*, as Alg; similarly, for $k=\infty$, we shall denote $\mathrm{Alg}_{\mathbb{E}_\infty}$, the ∞*-category of commutative ring spectra*, as CAlg.

Remark 3.5.7. Let R be a *discrete* $\mathbb{E}_k$-ring spectrum, i.e., $\pi_i R\cong 0$ for any $i\neq 0$. Then R belongs to the heart of the t-structure on Sp of Proposition 3.3.12, which is equivalent to the ordinary abelian category Ab of abelian groups. In this case, R can be regarded as an $\mathbb{E}_k$-ring object in Ab: for $k=1$ this means that R is an ordinary associative ring, while for any $k\geqslant 2$ this means that R is an ordinary commutative ring. On the converse, any ordinary associative or commutative ring can be considered as a *discrete* $\mathbb{E}_1$- or $\mathbb{E}_\infty$-ring spectrum, respectively: this means that the theory of ring spectra contains properly the theory of ordinary rings from classical algebra. Notice that at the level of discrete rings, either a ring is commutative or it is not: indeed, if $\mathscr{C}$ is an n-category (i.e., it has only trivial k-morphisms for $k > n$), then $\mathbb{E}_k$-algebras are naturally equivalent to $\mathbb{E}_\infty$-algebras for any $k \geqslant n+1$ ([Lur17, Corollary 5.1.1.7]). Conceptually this is motivated as follows: for any $k > n$, we only have trivial k-simplices producing homotopies between $(k-1)$-morphisms, hence either there is a k-simplex testifying to the $(k-1)$-commutativity, or there is not. In particular, k-fold commutativity becomes a *property*, and not a *structure*.

The discussion carried out in Remark 3.4.24 yields an ∞-functor

$$\mathrm{LMod}_{\_}\colon \mathrm{Alg} \longrightarrow \widehat{\mathrm{Cat}}_\infty$$

sending an associative ring spectrum R to its ∞-category of left R-modules in spectra $\mathrm{LMod}_R := \mathrm{LMod}_R(\mathrm{Sp})$; an analogous claim holds also for the ∞-functor $\mathrm{RMod}\colon \mathrm{Alg} \to \widehat{\mathrm{Cat}}_\infty$. Actually, for any $k \geqslant 1$ and for any $\mathbb{E}_k$-ring spectrum R, the ∞-categories LMod_R and RMod_R are *stable* ([Lur17, Corollary 7.1.1.5]) and *presentable* ([Lur17, Corollary 4.2.3.7]): this follows from the fact that for any symmetric monoidal ∞-category $\mathscr{C}$. It turns out that it is quite simple to determine which stable and presentable ∞-categories arise in this way.

Theorem 3.5.8 ([Lur17, Theorems 4.8.5.16 and 7.1.2.1 and Proposition 7.1.2.6])**.** *Let* $\mathrm{Alg}^\wedge$ *be the ∞-category of associative ring spectra, endowed with the symmetric monoidal structure provided by Example 3.4.13.(5), and let*

$$\left(\mathrm{Stab}^{\mathrm{pr}}_\infty\right)^{\otimes}_{\mathrm{Sp}/} := \mathrm{Mod}_{\mathrm{Sp}}\left(\mathrm{Pr}^{\mathrm{L}}\right)^{\otimes}_{\mathrm{Sp}/}$$

be the ∞-category of stable presentable ∞-categories, endowed with the symmetric monoidal structure induced by the on Pr^{L} described in Theorem 3.5.2[6]*. Then the ∞-functor*

$$\mathrm{LMod}_{\_}\colon \mathrm{Alg} \longrightarrow \left(\mathrm{Stab}^{\mathrm{pr}}_\infty\right)^{\otimes}_{\mathrm{Sp}/} \tag{3.5.9}$$

is a fully faithful, colimit-preserving and strongly monoidal ∞-functor. In particular, using Theorem 3.4.28, one has a fully faithful embedding

$$\mathrm{LMod}_{\_}\colon \mathrm{Alg}_{\mathbb{E}_k} \longrightarrow \mathrm{Alg}_{\mathbb{E}_{k-1}}\left(\mathrm{Mod}_{\mathrm{Sp}}\left(\mathrm{Pr}^{\mathrm{L}}\right)\right). \tag{3.5.10}$$

A presentable and stable ∞-category $\mathscr{C}$ belongs to the essential image of the embedding 3.5.9 if and only if the object $F(\mathbb{S})$ determined by the pointing $F\colon \mathrm{Sp} \to \mathscr{C}$ is a compact generator of $\mathscr{C}$, i.e., if and only if the ∞-functor

$$\underline{\mathrm{Map}}_{\mathscr{C}}(F(\mathbb{S}), -)\colon \mathscr{C} \longrightarrow \mathrm{Sp}$$

is a conservative ∞-functor which commutes with filtered colimits. If moreover $\mathscr{C}$ is equipped with an $\mathbb{E}_{k-1}$-monoidal structure compatible with colimits separately in each variable, then it belongs to the essential image of the embedding 3.5.10 if and only if the unit $\mathbb{1}_{\mathscr{C}}$ is a compact generator in the above sense.

[6] Notice that if $\mathscr{C}$ and $\mathscr{D}$ are presentable and stable, then $\mathscr{C} \otimes \mathscr{D}$ is again presentable and stable. The only caveat is that the unit is Sp instead of $\mathcal{S}$.

Warning 3.5.11. For any stable and presentable ∞-category $\mathscr{C}$, given a compact generator C one has an equivalence of stable and presentable ∞-categories

$$\mathscr{C} \simeq \mathrm{LMod}_{\underline{\mathrm{Map}}_{\mathscr{C}}(C,C)}.$$

In particular, if $\mathscr{C} \simeq \mathrm{LMod}_R$ for an associative ring spectrum R, *any* other compact generator C yields a Morita equivalence of stable and presentable ∞-categories

$$\mathrm{LMod}_R \simeq \mathrm{LMod}_{\underline{\mathrm{Map}}_R(C,C)},$$

where $\underline{\mathrm{Map}}_R(C,C)$ is seen as an associative ring spectrum via composition of maps. In particular, R and $\underline{\mathrm{Map}}_R(C,C)$ can be *very different* one from the other: for example, if $R = \mathrm{H}\Bbbk$ is the Eilenberg-Maclane ring spectrum associated to a field, then $C := \mathrm{H}\Bbbk^n$ is a compact generator of the ∞-category of $\mathrm{H}\Bbbk$-modules in the above sense for any integer n, and in particular one has an equivalence of ∞-categories

$$\mathrm{Mod}_{\mathrm{H}\Bbbk} \simeq \mathrm{LMod}_{\mathrm{H}\,\mathrm{M}_{n\times n}(\Bbbk)}.$$

This is why it is crucial, in the hypothesis of Theorem 3.5.8, that our ∞-category is *pointed* by a colimit preserving ∞-functor $F\colon \mathrm{Sp} \to \mathscr{C}$: in the above situation, without the pointing $\mathrm{Sp} \to \mathscr{C}$, which is completely determined by the image of the sphere spectrum $\mathbb{S}$, we would not be able to distinguish between LMod_R and $\mathrm{LMod}_{\underline{\mathrm{Map}}_R(C,C)}$, hence the ∞-functor would be far from being a fully faithful embedding.

Construction 3.5.12. In the limiting case $k = \infty$, Theorem 3.5.8 says that for a commutative ring spectrum R the ∞-category

$$\mathrm{Mod}_R := \mathrm{LMod}_R \simeq \mathrm{RMod}_R$$

is a stable and presentable ∞-category which is moreover symmetric monoidal. The symmetric monoidal structure on Mod_R is provided by the *relative tensor product* over R, which generalizes the derived tensor product over ordinary commutative rings. Given two R-modules M and N, we define their relative tensor product over R via a *bar construction*, i.e., by taking the geometric realization of the simplicial R-bimodule

$$M \otimes_R N := \operatorname*{colim}_{n\in\mathbb{N}}\Big(\dots \begin{smallmatrix}\rightarrow\\[-2pt]\leftarrow\\[-2pt]\rightarrow\\[-2pt]\leftarrow\\[-2pt]\rightarrow\\[-2pt]\leftarrow\\[-2pt]\rightarrow\end{smallmatrix} M \wedge R^{\wedge 3} \wedge N \begin{smallmatrix}\rightarrow\\[-2pt]\leftarrow\\[-2pt]\rightarrow\\[-2pt]\leftarrow\\[-2pt]\rightarrow\end{smallmatrix} M \wedge R^{\wedge 2} \times N \begin{smallmatrix}\rightarrow\\[-2pt]\leftarrow\\[-2pt]\rightarrow\end{smallmatrix} M \wedge R \wedge N \begin{smallmatrix}\leftarrow\\[-2pt]\rightarrow\end{smallmatrix} M \wedge N\Big).$$

In the simplicial R-bimodule above, the face maps

$$M \wedge R^{\wedge n} \wedge N \longrightarrow M \wedge R^{\wedge(n-1)} \wedge N$$

encode the actions of the first copy of R over M, of the last copy of R over n, and all possible multiplications between the i-th and the $(i+1)$-th copies of R, for i ranging from 1 to $n-1$; on the other hand, the degeneracy maps

$$M \wedge R^{\wedge(n-1)} \wedge N \longrightarrow M \wedge R^{\wedge n} \wedge N$$

are described by inserting the unit $\mathbb{1}_R$ in the i-th position, for i ranging from 1 to n. This is the core content of the, otherwise *very* technical, [Lur17, Section 4.4.2].

The relative tensor product over a commutative ring spectrum R, just like the smash product, commutes with colimits separately in each variable and admits R as its unit. Moreover, it is associative up to *canonical* homotopy: for two commutative ring spectra R and S, given an R-module M, an S-module N and an (R,S)-bimodule P one has a contractible space of equivalences between $(M\otimes_R P)\otimes_S N$ and $M\otimes_R(P\otimes_S N)$. This implies that for any commutative ring spectrum R, and for any R-module S which is a commutative algebra for the relative tensor product over R, one has equivalences of symmetric monoidal ∞-categories

$$\mathrm{Mod}_S(\mathrm{Mod}_R)\simeq \mathrm{Mod}_S.$$

Remark 3.5.13. Let R and S be commutative ring spectra, and suppose we have a map of commutative ring spectra $f\colon R\to S$. The discussion of Remark 3.4.24 provides a forgetful ∞-functor

$$\mathrm{oblv}_S := f^*\colon \mathrm{Mod}_S \longrightarrow \mathrm{Mod}_R$$

which in virtue of Proposition 3.4.21 is an ∞-functor between presentable ∞-categories which commutes with all limits and colimits. In particular, it admits two adjoint.

(1) The *left adjoint*

$$-\otimes_R S\colon \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_S$$

is the *base change ∞-functor*. At the level of the underlying spectra, this ∞-functor is precisely described by sending an R-module M to the relative tensor product $M\otimes_R S$ defined in Construction 3.5.12, where we view S as an (R,S)-bimodule via f.

(2) The *right adjoint*

$$\underline{\mathrm{Map}}_R(S,-)\colon \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_S$$

takes an R-module M to the mapping spectrum of R-linear maps $\underline{\mathrm{Map}}_R(S,M)$, which admits an S-module structure informally described by acting with scalars in S on the domain of a map $\varphi\colon S\to M$.

The canonical associativity of the relative tensor product assures that the left adjoint is strongly monoidal with respect to the relative tensor product on both Mod_R and Mod_S.

The symmetric monoidal structure on Mod_R given by the relative tensor product of Construction 3.5.12 allows us to consider $\mathbb{E}_k$-algebra objects in it, and this motivates the following.

Definition 3.5.14. Let R be a commutative ring spectrum, and let $\mathrm{Mod}_R^{\otimes}$ be the ∞-category of R-modules in spectra endowed with the relative tensor product. An *R-$\mathbb{E}_k$-algebra* (or *$\mathbb{E}_k$-algebra over R*) is an $\mathbb{E}_k$-algebra in $\mathrm{Mod}_R^{\otimes}$. We denote the ∞-category of R-$\mathbb{E}_k$-algebras as $R\text{-}\mathrm{Alg}_{\mathbb{E}_k}$.

Notation 3.5.15. Again, if $k=1$ or $k=\infty$, an R-$\mathbb{E}_k$-algebra is simply an associative or commutative R-algebra, respectively. In this case, we shall adopt the notations R-Alg and R-CAlg, or more frequently the notations Alg_R and CAlg_R, to denote the ∞-categories of $\mathbb{E}_1$- and $\mathbb{E}_\infty$-algebras over R, respectively.
If moreover $R=:\Bbbk$ is a discrete commutative ring, we shall compactify notations and write $\mathrm{Alg}_{\Bbbk}$ and $\mathrm{CAlg}_{\Bbbk}$ instead of $\mathrm{Alg}_{\mathrm{H}\Bbbk}$ and $\mathrm{CAlg}_{\mathrm{H}\Bbbk}$.

Remark 3.5.16.

(1) The discussion of Remark 3.5.13 guarantees that the adjunction

$$-\otimes_R S\colon \mathrm{Mod}_R \rightleftarrows \mathrm{Mod}_S\colon \mathrm{oblv}_S$$

lifts to an adjunction at the level of $\mathbb{E}_k$-algebra objects for any $k \geqslant 0$, because the left adjoint ∞-functors is strongly monoidal, hence the right adjoint is lax monoidal, and both preserve algebra objects for any ∞-operad. In particular, we have an adjunction

$$- \otimes_R S \colon R\text{-}\mathrm{Alg}_{\mathbb{E}_k} \rightleftarrows S\text{-}\mathrm{Alg}_{\mathbb{E}_k} \colon \mathrm{oblv}_S \,.$$

(2) In ordinary commutative algebra, we can see a commutative R-algebra S either as a commutative ring object in the category of R-modules with respect to the relative tensor product, or as a commutative ring equipped with a ring map $\varphi \colon R \to S$. This holds as well in the homotopical setting ([Lur17, Corollary 3.4.1.7]): we have an equivalence of ∞-categories

$$\mathrm{CAlg}_R \coloneqq \mathrm{CAlg}(\mathrm{Mod}_R) \simeq (\mathrm{CAlg})_{R/}.$$

Under this equivalence, the forgetful ∞-functor $\mathrm{oblv}_S \colon \mathrm{CAlg}_S \to \mathrm{CAlg}_R$ agrees with the pullback ∞-functor φ^*.

We conclude this chapter, and these notes, with some results about *rigidification* of algebras, commutative algebras, and modules in the ∞-category of spectra. First, let us remark the following fact. Let $\mathscr{M}^{\otimes}$ be a model category equipped with a (symmetric) monoidal structure $\otimes$ sufficiently compatible with the model structure, i.e., $\mathbb{1}_{\mathscr{M}}$ is cofibrant, the monoidal structure is closed, and

$$- \otimes - \colon \mathscr{M} \times \mathscr{M} \longrightarrow \mathscr{M}$$

is a Quillen bifunctor. Then, its hammock localization (Definition 2.3.1) $\mathscr{C} \coloneqq \mathrm{L}^{\mathrm{H}}(\mathscr{M}, \mathscr{W})$ is a (symmetric) monoidal ∞-category, and the monoidal structure is induced by the one on $\mathscr{M}$ by taking the *derived* tensor product ([Lur17, Section 4.7.1]). We want to be able to answer to the following questions.

Question 3.5.17.

(1) Let A be an associative algebra in $\mathscr{C}^{\otimes}$. Is A equivalent to a strictly unital and associative algebra $\underline{A}$ in $\mathscr{M}^{\otimes}$?

(2) Let A be a commutative algebra in $\mathscr{C}^{\otimes}$. Is A equivalent to a strictly unital and commutative algebra $\underline{A}$ in $\mathscr{M}^{\otimes}$?

(3) Recall that the Eilenberg-Maclane spectrum ∞-functor (Remark 3.5.7) provides a way to view ordinary (commutative) rings as discrete (commutative) ring spectra. In particular, given an ordinary ring $\Bbbk$ we can either consider the stable derived ∞-category $\mathscr{D}(\Bbbk)$ of left $\Bbbk$-modules, or the stable ∞-category $\mathrm{LMod}_{\mathrm{H}\Bbbk}$ of left $\mathrm{H}\Bbbk$-modules in the ∞-category of spectra. How are these two objects related?

It turns out that we can answer pretty neatly to *all* of the above. We shall start from Question 3.5.17.(3)

Theorem 3.5.18 ([Lur17, Sections 7.1.1 and 7.1.2])**.** *Let R be a ring spectrum, and let LMod_R be the ∞-category of left R-modules in spectra. Then LMod_R is equipped with a t-structure, where connective objects are those modules whose stable homotopy groups are 0 in negative degrees*[7]*, and*

[7] If R is itself connective as a spectrum, then coconnective objects are indeed modules whose stable homotopy groups are 0 in positive degrees; however, in general this is *not* true!

$\mathrm{LMod}_R^{\heartsuit}$ *is equivalent to the ordinary abelian category of left* $\pi_0 R$*-modules. The universal property of the bounded stable derived* ∞*-category* $\mathcal{D}^-(\pi_0 R)$ *(Theorem 3.2.10) provides an* ∞*-functor*

$$\mathcal{D}^-(\pi_0 R) \longrightarrow \mathrm{LMod}_R$$

which is fully faithful if and only if R *is discrete. In particular, in this case one has* $R \simeq \pi_0 R$ *and an equivalence of* ∞*-categories*

$$\mathcal{D}(R) \simeq \mathrm{LMod}_R.$$

If R *is discrete and commutative, then the above equivalence is strongly monoidal.*

Notation 3.5.19. Theorem 3.5.18 implies in particular that for a discrete commutative ring $\Bbbk$ the stable/dg derived ∞-category of $\Bbbk$-modules is equivalent to the ∞-category of $\mathrm{H}\Bbbk$-modules, and that the relative tensor product of $\mathrm{H}\Bbbk$-modules agrees with the usual derived tensor product of $\Bbbk$-modules. This justifies the choice of denoting the stable derived ∞-category of $\Bbbk$-modules simply as $\mathrm{Mod}_\Bbbk$, which is widely adopted in derived algebraic geometry.

Remark 3.5.20 (Characterization of perfect complexes in homotopical algebra, [Lur17, Proposition 7.2.4.4])**.** In ordinary homological algebra, one defines *perfect complexes of* $\Bbbk$*-modules* as those objects in the (triangulated) derived category of $\Bbbk$ which are quasi-isomorphic to bounded complexes of finitely generated and projective $\Bbbk$-modules. Under the equivalence provided by Theorem 3.5.18, we can characterize the ∞-category $\mathrm{Perf}_\Bbbk$ of perfect $\Bbbk$-modules either as the smallest stable sub-∞-category of $\mathrm{Mod}_\Bbbk$ containing $\Bbbk$ and closed under finite colimits, or as the full sub-∞-category of $\mathrm{Mod}_\Bbbk$ over which the $\Bbbk$-linear duality ∞-functor

$$\underline{\mathrm{Map}}_\Bbbk(-, \Bbbk)\colon \mathrm{Perf}_\Bbbk \longrightarrow \mathrm{Perf}_\Bbbk^{\mathrm{op}}$$

is an equivalence. Alternatively, $\mathrm{Perf}_\Bbbk$ is the full sub-∞-category of $\mathrm{Mod}_\Bbbk$ spanned by *compact objects*, i.e., objects M for which the ∞-functor

$$\mathrm{Map}_\Bbbk(M, -)\colon \mathrm{Mod}_\Bbbk \longrightarrow \mathcal{S}$$

commutes with filtered colimits.

Next, let us focus on Question 3.5.17.(1). While this result can be heavily refined in more general terms for a large class of monoidal model categories, we shall consider $\mathcal{M}^\otimes$ to be either the model category $\mathrm{C}_\bullet(\Bbbk)^\otimes$ of chain complexes over a base commutative ring $\Bbbk$ equipped with the Künneth tensor product, or the model category $\mathrm{sMod}_\Bbbk^\otimes$ of simplicial $\Bbbk$-modules with degree-wise tensor product.

Theorem 3.5.21 ([Lur17, Proposition 4.1.8.3 and Theorem 4.1.8.4])**.** *Let* $\mathcal{M}^\otimes$ *be as above. Then the category of associative algebras in* $\mathcal{M}^\otimes$ *admits a combinatorial model structure, where weak equivalences and fibrations are detected by the forgetful functor*

$$\mathrm{oblv}_{\mathrm{Alg}}\colon \mathrm{Alg}(\mathcal{M}) \longrightarrow \mathcal{M}.$$

In particular, the forgetful functor becomes a right Quillen functor.
Taking the hammock localization of such model structure, one has an equivalence of ∞*-categories*

$$\mathrm{L}^{\mathrm{H}}(\mathrm{Alg}(\mathcal{M}), \mathcal{W}) \simeq \mathrm{Alg}\big(\mathrm{L}^{\mathrm{H}}(\mathcal{M}, \mathcal{W})\big).$$

Remark 3.5.22. Let us spell out more concretely the consequences of Theorem 3.5.21. Let $\mathcal{M} := \mathrm{C}_\bullet(\Bbbk)$ be the symmetric monoidal model category of chain complexes of $\Bbbk$-modules. An algebra $A_\bullet$ in $\mathrm{C}_\bullet(\Bbbk)$ is a chain complex endowed with a collection of graded bilinear multiplications

$$\cdot\colon A_p \otimes_{\Bbbk} A_q \longrightarrow A_{p+q}$$

such that the differential is a graded derivation for this product, i.e.,

$$\mathrm{d}(a\cdot b) = \mathrm{d}(a)\cdot b + (-1)^{|a|} a \cdot \mathrm{d}(b).$$

Let $\mathrm{dgAlg}_{\Bbbk}$ be the category of differential graded algebras over $\Bbbk$,. Then Theorem 3.5.21 tells us that $\mathrm{dgAlg}_{\Bbbk}$ is model category where weak equivalences are quasi-isomorphisms and fibrations are surjections in each degree (Theorem 1.3.11), and we have an equivalence of ∞-categories

$$\mathrm{L}^{\mathrm{H}}\left(\mathrm{dgAlg}_{\Bbbk}, \mathcal{W}\right) \simeq \mathrm{Alg}\left(\mathrm{L}^{\mathrm{H}}(\mathrm{C}_\bullet(\Bbbk), \mathcal{W})\right).$$

The right hand side in the above equivalence is actually equivalent to the dg enhancement of the abelian $\Bbbk$-linear category $\mathrm{C}_\bullet(\Bbbk)$ discussed in Construction 3.2.5, which is a model for the stable derived ∞-category of $\Bbbk$. In particular, we have a chain of equivalences

$$\mathrm{L}^{\mathrm{H}}\left(\mathrm{dgAlg}_{\Bbbk}, \mathcal{W}\right) \simeq \mathrm{Alg}\left(\mathrm{L}^{\mathrm{H}}(\mathrm{C}_\bullet(\Bbbk), \mathcal{W})\right) \simeq \mathrm{Alg}(\mathcal{D}(\Bbbk)) \overset{\text{Thm } 3.5.18}{\simeq} \mathrm{Alg}(\mathrm{Mod}_{\Bbbk}).$$

This tells us the following: *$\Bbbk$-$\mathbb{E}_1$-algebras in spectra can be rectified to strict unital and associative differential graded $\Bbbk$-algebras*.

We finally deal with Question 3.5.17.(2). Differently from the case of merely associative algebras, the answer to this problem is *strictly weaker*: we have to restrict ourselves to consider only those $\mathbb{E}_\infty$-ring spectra which are $\mathrm{H}\mathbb{Q}$-algebras. The core problem is that in order to have an analogous result to Theorem 3.5.21 we need some stronger compatibility with the monoidal structure on a model category $\mathcal{M}^\otimes$ and the class of its cofibrations – namely, we want every n-th symmetric power of a cofibration to be a cofibration as well in the model category of objects of $\mathcal{M}$ equipped with an action of the symmetric group Σ_n. This is an algebraic/category-theoretical property that yields a *free action* of Σ_n on an n-fold tensor product $X^{\otimes n}$

Theorem 3.5.23. *Let $\Bbbk$ be an ordinary commutative $\Bbbk$-algebra, and let $\mathcal{M}^\otimes$ be either the category of chain complexes over $\Bbbk$ equipped with the Künneth tensor product, or the category of simplicial $\Bbbk$-modules equipped with the degree-wise tensor product. Then the category of commutative algebras in $\mathcal{M}^\otimes$ admits a combinatorial model structure, where weak equivalences and fibrations are detected by the forgetful functor*

$$\mathrm{oblv}_{\mathrm{CAlg}}\colon \mathrm{CAlg}(\mathcal{M}) \longrightarrow \mathcal{M}.$$

In particular, the forgetful functor becomes a right Quillen functor.
Taking the hammock localization of such model structure, one has an equivalence of ∞-categories

$$\mathrm{L}^{\mathrm{H}}(\mathrm{CAlg}(\mathcal{M}), \mathcal{W}) \simeq \mathrm{CAlg}\left(\mathrm{L}^{\mathrm{H}}(\mathcal{M}, \mathcal{W})\right).$$

Remark 3.5.24. A commutative algebra $A_\bullet$ in $\mathrm{C}_\bullet(\Bbbk)$ is called *commutative differential graded $\Bbbk$-algebra*. Unraveling the definitions, we see that a commutative differential graded $\Bbbk$-algebra is just a differential graded $\Bbbk$-algebra which is moreover *graded-commutative*, i.e.,

$$a\cdot b = (-1)^{|a|\cdot|b|} b\cdot a,$$

where the sign comes from the fact that the symmetry isomorphism between $C_\bullet \otimes_{\Bbbk} D_\bullet$ and $D_\bullet \otimes_{\Bbbk} C_\bullet$ is given by sending an homogeneous element $c \otimes d$ of degree $|c| \cdot |d|$ to $(-1)^{|c|\cdot|d|} d \otimes C$. Arguing as in Remark 3.5.22, Remark 3.5.22 yields that *if* $\Bbbk$ *contains the field of rational numbers* then the category $\mathrm{cdga}_{\Bbbk}$ of differential graded commutative $\Bbbk$-algebras admits a model structure (where again weak equivalences are quasi-isomorphisms and fibrations are surjections in every degree), and $\Bbbk$-$\mathbb{E}_\infty$-algebras in spectra are modeled by commutative differential graded $\Bbbk$-algebras, i.e.,

$$\mathrm{CAlg}_{\Bbbk} \simeq \mathrm{L}^{\mathrm{H}}\left(\mathrm{cdga}_{\Bbbk}, \mathscr{W}\right).$$

Warning 3.5.25 ([Lur17, Warning 7.1.4.21])**.** When $\Bbbk$ is not a $\mathbb{Q}$-algebra, the category of commutative differential graded algebras over $\Bbbk$ *cannot* be endowed with a model structure turning the forgetful functor to chain complexes into a right Quillen functor. On the other hand, simplicial commutative $\Bbbk$-algebras (i.e., simplicial objects in the category of commutative $\Bbbk$-algebras) can *always* be equipped with a model structure defined as in Theorem 3.5.23: weak equivalences and fibrations are detected by the forgetful functor to simplicial $\Bbbk$-modules, turning it into a right Quillen functor. There exists an ∞-functor

$$\mathrm{L}^{\mathrm{H}}\left(\mathrm{sCAlg}_{\Bbbk}, \mathscr{W}\right) \longrightarrow \mathrm{CAlg}_{\Bbbk} := \mathrm{CAlg}(\mathrm{Mod}_{\Bbbk})$$

but it is not an equivalence if $\Bbbk$ is not a $\mathbb{Q}$-algebra.

3.6. **Some exercises.**

(1) Prove some of the exercises and statements left unproved in this section. (Hint: if the statement is a theorem, *usually* it is quite difficult to do it on your own.)
(2) Let $\mathscr{C}_{\geqslant 0}$ be a full presentable sub-∞-category of a presentable stable ∞-caategory $\mathscr{C}$ which is stable under colimits. Prove that there exists a t-structure on $\mathscr{C}$ whose ∞-category of connective objects coincides with $\mathscr{C}_{\geqslant 0}$, and for an object X in $\mathscr{C}$ write explicitly how the fiber sequence
$$\tau_{\geqslant 0} X \longrightarrow X \longrightarrow \tau_{\leqslant -1} X$$
is defined.
(3) Let $\Bbbk$ be an ordinary commutative ring of characteristic 0, and let M a discrete free $\Bbbk$-module in the usual sense. What is $\mathrm{Sym}_{\Bbbk}(M[1])$?
(4) Prove that an ordinary discrete commutative ring R is finitely presented if and only if the functor
$$\mathrm{Hom}_{\mathrm{CAlg}_R^{\heartsuit}}(R, -)\colon \mathrm{CAlg}_R^{\heartsuit} \longrightarrow \mathrm{Set}$$
commutes with filtered colimits.
(5) Let M be a $\Bbbk$-module, for $\Bbbk$ an $\mathbb{E}_\infty$-ring spectrum, and let $\mathrm{Sym}_{\Bbbk}\colon \mathrm{Mod}_{\Bbbk} \longrightarrow \mathrm{CAlg}_{\Bbbk}$ denote the free commutative $\Bbbk$-$\mathbb{E}_\infty$-ring spectrum ∞-functor. Show that
$$\begin{tikzcd} \mathrm{Sym}_{\Bbbk}(M) \arrow[r, "f"] \arrow[d, "f"'] & \Bbbk \arrow[d, "g"] \\ \Bbbk \arrow[r, "g"'] & \mathrm{Sym}_{\Bbbk}(M[1]) \end{tikzcd}$$

is a pushout in the ∞-category of $\mathrm{CAlg}_{\Bbbk}$, where f is induced by the map of $\Bbbk$-modules $M \to 0$ and where g is the structure morphism.

(6) Let $\Bbbk[\varepsilon]$ the free commutative $\Bbbk$-algebra on $\Bbbk[1]$. Show that $\Bbbk[\varepsilon]$ is equivalent to the square-zero extension $\Bbbk \oplus \Bbbk[1]$. Prove that $\Bbbk$ is not a perfect $\Bbbk[\varepsilon]$-module.

(7) Prove that for a commutative ring spectrum $\Bbbk$ the forgetful ∞-functor $\mathrm{CAlg}_{\Bbbk} \to \mathrm{Alg}_{\Bbbk}$ is not fully faithful (Warning 3.4.27) with the following example. Let $\Bbbk$ be an ordinary commutative ring containing the field $\mathbb{Q}$ of rational numbers, and let

$$\Bbbk[x, y] := \mathrm{Sym}_{\Bbbk}(\Bbbk \oplus \Bbbk)$$

be the free commutative $\Bbbk$-algebra in two variables. Let $\mathrm{oblv}_{\mathrm{CAlg}} \colon \mathrm{CAlg}_{\Bbbk} \longrightarrow \mathrm{Alg}_{\Bbbk}$ be the forgetful ∞-functor from the ∞-category of commutative $\Bbbk$-algebra spectra to the ∞-category of associative $\Bbbk$-algebra spectra. Let A be the commutative $\Bbbk$-algebra spectrum given by $\Bbbk$ in homological degrees 0 and 1, with square zero multiplication. Show that

$$\mathrm{Map}_{\mathrm{CAlg}_{\Bbbk}}(\Bbbk[x, y], A), \not\simeq \mathrm{Map}_{\mathrm{Alg}_{\Bbbk}}\big(\mathrm{oblv}_{\mathrm{CAlg}}(\Bbbk[x, y])\ \mathrm{oblv}_{\mathrm{CAlg}}(A)\big).$$

(Hint: use the model structures on differential graded algebras of Theorems 3.5.21 and 3.5.23.)

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